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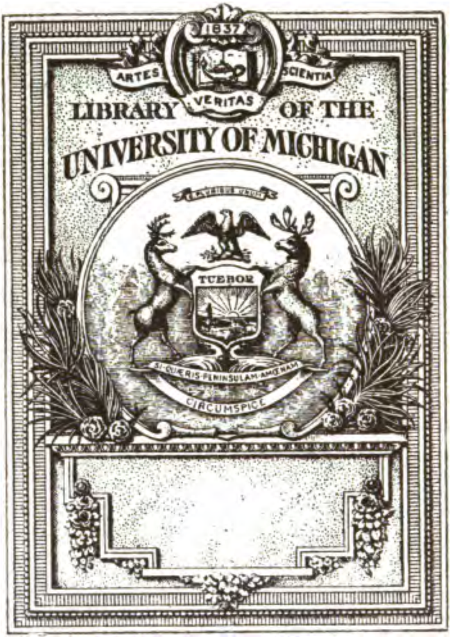

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A TREATISE ON THE CALCULUS OF VARIATIONS.

BY

RICHARD ABBATT, F.R.A.S., &c.

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PREFACE.

Since the publication of Professor Woodhouse's *Treatise on Isoperimetrical Problems* in 1810, no work has appeared in English on the Calculus of Variations which can be said to embrace the subject generally; and the student is thus left to seek information from foreign authors.

In Lacroix's *Traité du Calcul Différentiel et du Calcul Intégral,* Tom. II., we find materials sufficient to form a complete work on Variations; but the subject is treated in a manner so prolix and inelegant, that the reader's taste will scarcely be improved, how much soever his knowledge may be increased by the perusal. The want of examples also for illustrating the theory, must add greatly to the labour of the student, and retard his progress both in this and in many of the other subjects treated in the author's three ponderous volumes.

In a branch of analytical science so abstract as the Calculus of Variations, which originated in solving a particular class of problems, and gradually reached its present perfect state, little progress can be made without constant reference to examples; and probably no student who turns his attention for the first time to the subject of Variations, will fully understand the articles 16—22, in the following work, until he has applied them to particular instances. I have, therefore, introduced many examples, as the only means of lessening to my readers the difficulties of so abstract a theory.

The twenty-second and twenty-third chapters of Lagrange's *Leçons sur le Calcul des Fonctions*, contain a general theory of Variations, arranged with much clearness and elegance. But the illustrious geometer has rendered this part of his work less useful, and even in some degree, repulsive, to the generality of students, by rejecting the usual notation, and introducing new signs, which, having nothing to recommend them in preference to the old, have not been generally adopted. No one will dispute the right of Lagrange to model, in any manner he chose, the Calculus of Variations, in the invention of which he had so great a share; but those only will avail themselves of his speculations on this subject to whom different notations have become familiar, and a change of signs is a matter of little consequence.

Woodhouse's treatise is distinguished by great learning and depth of research. It contains a full and interesting history of those problems called isoperimetrical, which formerly occupied so much of the attention of mathematicians. The reader will also see beautiful inductive analyses of two general theorems, which Euler, with singular patience and assiduity, arrived at, and which are of the utmost use in the solution of these problems, or rather may be said completely to solve them. The contemplation of such instances cannot fail to be interesting to those who are curious to observe how particular solutions and indirect and complicated processes give rise to simple and general theories.

The following treatise has been written principally with the view of rendering a difficult and abstruse subject more easy and accessible, and of supplying the place of Woodhouse's problems in the course of study pursued at our universities; for the short piece on Variations in the *Mathematical Tracts* of the Astronomer Royal, can hardly be said to have superseded its necessity. And the student who has made himself acquainted with the Differential and Integral Calculus, will find abundant exercise, of the most attractive kind, in the Calculus of Variations, affording, as it does, the choicest examples of integration.

RICHARD ABBATT.

EPPING, 1837.

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T a function of

$$x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, \text{&c.}, u_{,};\ u_{,} = \int T_{,}\,dx:$$

$T_{,}$ a function of

$$x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, \text{&c.}, u_{,,};\ u_{,,} = \int T_{,,}\,dx:$$

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CALCULUS OF VARIATIONS.

SECTION I.

1. WHEN a variable quantity increases to a finite magnitude and then decreases, this magnitude, in reference to the variable quantity, is said to be a *maximum;* and if a quantity decreases to a certain value, and afterwards increases, this value is called its *minimum;* and it is only such quantities that admit of maximum and minimum. Thus, the product of the two parts into which any number may be divided, admits of a maximum, and the distance of a given point from a straight line admits of a minimum; whilst the distance of a given point from the periphery of an ellipse, is susceptible both of a maximum and minimum.

Problems of maxima and minima first occur in the Conics of Apollonius, in which nearly the whole of

Book V. treats of determining the greatest and least straight lines that can be drawn from given points upon the arcs of conic sections. The method of Apollonius consists simply in demonstrating that every other straight line drawn from the same point to the curve, is greater in the case of minimum, and less in the case of maximum, than that which he had determined; and this method, which was used by Euclid*, has been followed by all geometers who have been employed in solving problems of maxima and minima geometrically.

2. Fermat appears to have been the first that recognised the true principle which indicates the state of maximum or mininum, viz., when a quantity approaches either of these states, its variations become less and less, and at length vanish. In the application of this principle to the solutions of these problems, Fermat made use of a method which approaches so nearly to that of the Differential Calculus, that both Lagrange and Laplace† have considered Fermat as the real inventor. In his method *de maximis et de minimis*, he equated the function which is to be a maximum or minimum to the same expression, when the unknown or variable quantity is augmented or diminished by a small indetermi-

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* Book III. Prop. 7, 8.

† Leçons sur le Calcul des Fonctions, p. 321.
System of the World, Book V. Chap. 3.

nate quantity; then expunging the terms which are common to both sides, and dividing each of the remaining terms by the indeterminate quantity, he obtained an equation from which the value of the unknown quantity of the question might be deduced. The following example, which was given by Fermat, will illustrate his method.

To divide a straight line (a) into two such parts that their product may be the greatest possible.

Let x be one of the parts, then $a-x$ will be the other; and their product, $ax-x^2$ is to be a maximum. Adding to x the indeterminate quantity e, we shall have the new expression

$$a(x+e)-(x+e)^2;$$

whence we have

$$ax-x^2=a(x+e)-(x+e)^2;$$

expanding the terms of the second side, erasing those which are common to both, and dividing the remaining terms by e, we obtain

$$a-2x-e=0.$$

Now if we neglect e, as being 0, in the case of maximum, we shall have

$$a-2x=0,$$

$$\therefore x=\tfrac{1}{2}a;$$

shewing that the straight line must be divided into two equal parts, as we also know from Geometry. (Euc. II. 5.)

For another example, let us take that which requires the value of x, so that the expression x^3-bx^2 may be a maximum, which occurs in the problem to determine the greatest cylinder that can be cut from a given cone.

Here e being the arbitrary quantity as before, we have

$$x^3-bx^2=(x+e)^3-b(x+e)^2;$$

from which we deduce

$$0=3x^2-2bx+3ex+e^2-be,$$

or, omitting the terms affected with e,

$$3x^2-2bx=0,$$

$$\therefore x=\tfrac{2}{3}b.$$

In both of these examples we easily see that this method gives the same equation as that which would be obtained by equating the differential coefficient of the function, which is to be a maximum to 0, for the principle in both is the same, since the object in introducing the arbitrary quantity e, is merely to obtain the function which was afterward scalled the differential coefficient, or first derived function. Although the principle of the Differential Calculus is thus distinctly recognised in Fermat's method, yet, in its general application to problems, it is necessary to make radicals and transcendentals contained therein disappear, which require particular artifices; and these, it is precisely the object of the Differential

Calculus to obviate, by the simplicity and generality of its algorithm.

3. Problems of maxima and minima were among the first to which the Differential Calculus was applied.

Let

$$u = fx,$$

be the function which is to be a maximum or minimum, and let us suppose x to be replaced by $x+h$, then u' being the corresponding value of u, we shall have

$$u' - u = \frac{d\,.fx}{dx} \cdot \frac{h}{1} + \frac{d^2.fx}{dx^2} \cdot \frac{h^2}{1.2}$$

$$+ \frac{d^3.fx}{dx^3} \cdot \frac{h^3}{1.2.3} + \&c.$$

Now h may be taken so small that any term of this series will be greater than the sum of all the terms which follow, provided that none of the differential coefficients become infinite; in other words, that the development of the function is possible; so that the sign of the series will depend upon that of its first term. Hence when u is a maximum or minimum, $u'-u$, and consequently the first term of the series, $\frac{d.fx}{dx} \cdot \frac{h}{1}$, must be negative or positive respectively.

But since the factor h may be either positive or negative, and, being of the first degree, the sign of the term $\frac{d\,.\,f\,x}{d\,x}\,.\,\frac{h}{1}$ will depend upon that of h; therefore the increment $u'-u$ cannot be constantly positive or constantly negative, unless the first term of the series vanishes. Hence we have

$$\frac{d\,.\,f\,x}{d\,x}=0.$$

This condition being satisfied, the increment of u, or $u'-u$ becomes

$$\frac{d^2.\,f\,x}{d\,x^2}\,.\,\frac{h^2}{1\,.\,2}+\frac{d^3\,.\,f\,x}{d\,x^3}\,.\,\frac{h^3}{1\,.\,2\,.\,3}+,\ \&c.$$

which must still be negative when u is a maximum, and positive when u is a minimum; but the term $\frac{d^2.\,f\,x}{d\,x^2}\,.\,\frac{h^2}{1\,.\,2}$, having the square of h as a factor, will manifestly have its sign independent of that of h, and will therefore depend entirely upon the sign of the function $\frac{d^2.\,f\,x}{d\,x^2}$.

Hence, when $u=f\,x$ is a maximum or minimum, we shall have

$$\frac{d\,u}{d\,x}=0,$$

and to distinguish the one from the other, the quantity $\frac{d^2 u}{d x^2}$ will be negative in the former case and positive in the latter.

If both $\frac{d\,u}{d\,x}$ and $\frac{d^2\,u}{d\,x^2}$ vanish, then the same reasoning will shew that $\frac{d^3\,u}{d\,x^3}$, must also vanish, and $\frac{d^4\,u}{d\,x^4}$ be negative, when u is a maximum, and positive, when u is a minimum: and so on.

4. In the ordinary problems of maxima and minima in the Differential Calculus, and generally in all its applications, the nature of the function is supposed to be known, and to remain the same during the progress of the inquiry; and the object to be attained is the determination of that particular value of the variable which makes any proposed function a maximum or minimum. This, as has been explained in the preceding article, is accomplished by making the differential coefficient equal to 0, and an equation is obtained, the solution of which will give the value of the variable, corresponding to the maximum or minimum value of the function. Thus suppose the function $u = f\,x$ to be

$$u = \frac{a + x}{x}\sqrt{b^2 + x^2},$$

we have

$$\frac{d\,u}{d\,x} = \frac{x^3 - a\,b^2}{x^2\sqrt{b^2 + x^2}} = 0;$$

from which we obtain

$$x = \sqrt[3]{a\,b^2},$$

and the corresponding value of the function

$$u = \left(a^{\frac{2}{3}} + b^{\frac{2}{3}}\right)^{\frac{3}{2}},$$

is its minimum, since the second differential coefficient

$$\frac{d^2 u}{d x^2} = \frac{b^2 x^3 + 3 a b^2 x^2 + 2 a b^4}{(x \sqrt{b^2 + x^2})^3},$$

is a positive quantity.

There is then no difficulty in obtaining the general solution of problems of maxima and minima when the relation between the maximum or minimum quantity and the variable is known, since the subject is reduced to the solution of a finite equation.

But when the function is given only by means of an integration which cannot be performed in consequence of our ignorance of the nature of the functions contained in the differential coefficient, the ordinary operations of the Differential Calculus entirely fail, without the application of principles which are not usually explained in treatises on that branch of analysis. The general solution, however, has been accomplished by the aid of views which embrace more abstractedly the nature of functions. This theory, which we shall fully explain in the following pages, constitutes the Calculus of Variations.

SECTION II.

5. The Calculus of Variations, a name given by Euler to one of the first and principal discoveries of Lagrange, consists in a method of finding the functions relative to the maxima and minima of indeterminate integrals. As this subject is in a high degree abstract, it was only after the mind had long been familiarized with particular examples, that the general law which embraced them became gradually unfolded, and at length brought into view. Accordingly we find that James and John Bernoulli, and others, had made considerable progress in the solution of problems of this nature; when Euler, after following for some time their methods, undertook to generalize these particular solutions; and, viewing the subject more abstractly, he finally succeeded in reducing the various difficulties of this extensive problem into distinct classes, which required only the application of the Integral Calculus to complete the solutions. At last Lagrange, in accordance with views almost peculiar to his own mind, perceived that the subject was one of

pure analysis. By the invention of a distinct notation, he remodelled, in beautiful symmetry, all that had been effected by Euler, and by pointing out the meaning and use of what remains indeterminate at the limits, and other discoveries, he left in its present state this curious branch of analysis, which is only imperfect because the methods of the Integral Calculus are so.

6. When a given function of several variables and of their differential coefficients does not satisfy the conditions of integrability, it is not susceptible of integration unless there exist established relations among the variables referred to one which is regarded as the independent variable. The object then to be obtained is to determine such relations as will render the integral of the given function a maximum or minimum between given values of the variable, which is considered indeterminate.

For example, if V is a given function of x, y, $\frac{dy}{dx}$, $\frac{d^2 y}{dx^2}$, &c., let us suppose U to be the function of which V is the differential coefficient, so that $\frac{dU}{dx} = V.$

$$U = \int V\,dx;$$

and let $y = \varphi x$ be the function, which, by substitution in V, will render the quantity U, or $\int V\,dx$, between given limits, a maximum or minimum: the object of the Calculus is to determine the nature of the function $\varphi\, x$. Hence these problems of maxima and minima are of a much higher kind than those of the Differential Calculus; in consequence, however, of the attention which they received from geometers about the commencement of the last century, and the great interest excited in the minds of Euler and Lagrange, the solutions of particular problems, which were interesting in proportion to their difficulty, have become merged in that of the general problem which these illustrious geometers effected. The first problem of the kind on record* is the determination of the curve which generates the solid of least resistance in a fluid resisting as the square of the velocity; of which Newton gave a construction derived from the differential equation to the curve, without pointing out the method by which he obtained this equation. It was, however, supplied by Lagrange. (Cal. des Fonc., p. 425.) The next, though less difficult than the preceding, was the celebrated problem of the *brachystochrone,* or curve of quickest descent, which was proposed by John Bernoulli for solution in the Acta Eruditorum for

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* Principia, Lib. II. Prop. 34. Schol.

1696, published at Leipsic, under the following form:

PROBLEMA NOVUM,

Ad cujus solutionem mathematici invitantur.

Datis in plano verticali duobus punctis A et B, assignare mobili M, viam $A\ M\ B$ per quam gravitate sua descendens et moveri incipiens a puncto A, brevissimo tempore perveniat ad alterum punctum B. (Fig. 1.)

This problem was solved by James Bernoulli, Newton, and others, by methods founded upon the nature of a quantity in its state of minimum. Since the minimum property must be possessed by the whole curve, they assumed that any portion of it will possess the same property; and making two consecutive elements of the curve considered as a polygon of an infinite number of sides vary, they deduced, in terms of the co-ordinates of the angle formed by the elementary sides, the corresponding alteration in the minimum property, or time of descent, which, equated to 0, produces the differential equation to the curve. The difficulty in the solution of this problem, viewed analytically, consists in finding that particular function y which renders the integral $\int\sqrt{\left\{\frac{1+\left(\frac{dy}{dx}\right)^2}{y}\right\}}\,dx$, between given limits, a minimum. To effect this by the Differential Calculus, it must be integrated, and the differential coefficient $\sqrt{\left\{\frac{1+\left(\frac{dy}{dx}\right)^2}{y}\right\}}$ equated to 0; but the integration cannot be effected in consequence of the unknown nature of the function y, or the equation to the curve which it is the object of the inquiry to determine. But if we suppose the body M, at its departure from the point A, to be impressed with such a law of force, modified both in intensity and direction, as would constrain it to move to the point B in the shortest time, the path would be the curve required, and the law of the motion which was impressed upon the body at its commencement, expressed in terms of x, would determine the nature of the curve.

This problem, and its solutions, may be considered as the beginning of that long continued series of the labours of geometers, the object of which was the maxima and minima of indeterminate integrals. Soon after, to the property of maximum or minimum, another condition was added;—that the curve should be of a given length. Such as the problems, to find among all curves of equal perimeter, that which between given limits, shall contain the greatest or the least space; or the curve which by its revolution about its axis shall contain the greatest or least superficies, or which shall contain the greatest or least solidity. Or, to find the curve such, that constructing another curve upon the same axis, the ordinates of which are functions of the ordinates or arcs of the former, the area of the second curve shall be a maximum. And the difficulties of these problems, joined to the celebrity which attended the researches of the two brothers, James and John Bernoulli, Brook Taylor, and Euler, occasioned the general name of *isoperimetrical* to be given to all problems in which it is required to find curves that possess some property of maximum or minimum, with or without the condition of being of a given length [*].

When the curve is to possess the maximum or minimum property only, it is sufficient to make two elements vary, from which is deduced the variation of the maximum or minimum. When to the maximum or minimum property a given condition is added, it is necessary to make three consecutive elements or sides of the curve vary, in order to deduce two expressions in terms of the co-ordinates of the angles formed by the infinitely small sides;

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[*] Cal. des Fonc., p. 426.

one denoting the variation of the quantity which is to be a maximum or minimum, and the other that of the given length; these, separately equated to 0, produce two equations involving two arbitrary quantities, by the elimination of which is obtained the differential equation to the curve in which both the given and the maximum or minimum property will be incorporated. This was first done by James Bernoulli.

By the consideration of an infinitely small portion of a curve composed of two or three sides, the problems are reduced to the ordinary analysis of the Differential Calculus, and the difficulty consists in bringing the solutions to the differential form, by substituting for the abscissas and ordinates their values expressed by means of their infinitely small differences, which arise from the variation in position of the elements or sides of the curve; taking care to preserve terms of the same order, according to the law of homogeneity of infinitely small quantities. Upon these considerations were founded the solutions of the Bernoullis, Taylor, and the first solutions of Euler.

This last named mathematician took up the subject in the state in which it was left by his famous instructor, John Bernoulli; and, after the publication of various memoirs, he at length produced one of his most celebrated works, entitled, *Methodus* *inveniendi lineas curvas maximi minimique proprietate gaudentes*. This work, which appeared in 1744, contains a complete and general solution of isoperimetrical problems, as far as the nature of the curve is concerned, and is otherwise remarkable by the number and beauty of the examples which it contains. Without proceeding farther with the history, we shall refer the student to Lagrange's Leçons sur le Calcul des Fonctions, leçon xxi., and to Woodhouse's Treatise on Isoperimetrical Problems, in which he may trace the line of connection through the solutions of the Bernoullis and Taylor, the various memoirs and distinct treatise of Euler, and ending in the general principle, first distinctly perceived by Lagrange, requiring another view of differentiation *.

7. In publishing his new method, Lagrange made an important application to dynamics. Thus we may consider the different points of a body in motion, either by comparing, at the same instant, two consecutive points, or by comparing, at two consecutive instants, the positions of the same point. In the first case, the only relation which exists among the co-ordinates is expressed by the equation to the surface in which the points are situated; in the other, the co-ordinates vary according to the conditions of

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* Mémoires de l'Acad. de Turin, Tom. II. p. 174.

the established motion into which a new variable enters, which is the measure of the *time* or *force;* and these conditions determine the nature of the functions which express the varying co-ordinates of the point in motion. And he afterwards gave, in his *Mécanique Analytique,* many beautiful instances. From this consideration are solved problems in Dynamics, requiring the determination of a minimum by means of the principle of least action. Euler had availed himself of this principle, at the end of his treatise before mentioned, to determine the path of an isolated point; but Lagrange, by applying the principle of the conservation of living forces, extended the application to the motion of a system of bodies connected in any manner whatever; thus making it, in fact, the foundation of the science of Dynamics, instead of the principle of D'Alembert and the principle of virtual velocities. (Vid. *Mécanique Analytique,* Tom. I., pp. 246, 299, and Poisson's *Traité de Mécanique,* Tom. II., p. 2, and Tom. I., p. 654.)

A planet revolving about the sun describes an ellipse in space, of which the sun occupies one of the foci; and the co-ordinates of the planet, at any instant, may be rigorously calculated; but when the perturbating action of another planet is taken into account, these co-ordinates begin to vary according to another law, and become functions of the disturbing force and of the time. This mode of variation is precisely that to which the Calculus refers. Lastly, the greater part of the problems of Statics may be reduced to those requiring the determination of a maximum or minimum, to which we may apply the same principles. Thus the Calculus of Variations, the origin of which arose out of a question of pure curiosity, comprehends the whole of the applications of Mechanics and the most difficult problems of abstract Geometry.

SECTION III.

8. Having, in the preceding pages, given a brief history of Isoperimetrical problems and the Calculus of Variations, we shall, in this Section, proceed to point out more distinctly the nature of that change in a quantity which is termed its *variation*.

If we suppose y to be a function of x denoted by f, we shall have

$$y = f x;$$

and if we imagine the variable x to receive an increment h, the function y will receive a corresponding increment, which will entirely depend upon the nature of the function f, and the magnitude of the increment h; so that the equation

$$y = f x,$$

will become

$$y = f(x + h).$$

Subtracting the former from the latter and dividing by h, we shall have, for the ratio of the increments of y and x,

$$\frac{f(x + h) - f x}{h}.$$

Now, if h be taken indefinitely small, (not to vanish,) the first term of the development of the quantity

$$f(x+h)-fx,$$

is called the *differential* of y or fx; and the same term divided by h, or the first term of the expansion of the ratio

$$\frac{y'-y}{h}, \text{ or } \frac{f(x+h)-fx}{h},$$

is called the *differential coefficient*: the former is, for brevity, written dy, and the latter $\frac{dy}{dx}$*.

9. Resuming the equation

$$y=fx;$$

let us suppose in general, ~~that~~ the function fx, by some cause, to be changed into $f\,(x, i)$, so that the state of the equation

$$y=fx,$$

is, after the change, represented by

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* For a logical and very satisfactory explanation of the fundamental principles of the Differential and Integral Calculus, and of the application of these principles, the mathematical reader is referred to a Dissertation by W. S. B. Woolhouse, in the Appendix to the Gentleman's Diary, 1835, 1836.

$$y = f(x, i),$$

in which i is a new variable quantity introduced at the time.

Subtracting these two values, we have, for the increment of y, the expression

$$f(x, i) - fx.$$

Now if we suppose the function $f(x, i)$ to be developed according to the ascending powers of i, and remember that $f(x, i)$ becomes $f x$ when $i = 0$, we shall have, for the increment of y, the series

$$i \psi x + i^2 \psi' x + i^3 \psi'' x + \&c.,$$

in which ψx, $\psi' x$, &c., are functions of x, independent of i, the nature of which will entirely depend upon that of the function $f(x, i)$.

When i is infinitely small, the first term in the increment of y is called its *variation.*

Lagrange adopted the symbol δ to characterize this change in a quantity; so that dy representing the differential of y, δy will denote its variation; thus we shall have

$$\delta y = i \psi x;$$

hence the variation of y is an infinitely small arbitrary function of x, and consequently of y, on account of their primitive connection.

In general, we give the name of variations of the first order, to those terms of the total increment of y in which the infinitely small quantity i is of the first dimension; variations of the second order, those in which i rises to the second power, and so on.

If, in the function $f\,(x, i)$, we suppose x to be independent of the arbitrary quantity i, we shall have, by Maclaurin's theorem,

$$y = f\,x + \frac{d\,.\,f}{d\,i}\,.\,\frac{i}{1} + \frac{d^2\,.\,f}{d\,i^2}\,.\,\frac{i^2}{1\,.\,2}$$

$$+ \frac{d^3\,.\,f}{d\,i^3}\,.\,\frac{i^3}{1\,.\,2\,.\,3} + \&c.;$$

in which $f\,x, \frac{d\,.\,f}{d\,i}, \frac{d^2\,.\,f}{d\,i^2}$, &c., are the values of the function $f\,(x, i)$ and its successive differential coefficients taken with respect to i, when $i = 0$. Hence we have, in this case,

$$i\,\psi\,x = i\,.\,\frac{d\,.\,f}{d\,i} = \delta\,y,$$

$$i^2\,\psi'\,x = \frac{i^2}{2}\,.\,\frac{d^2\,.\,f}{d\,i^2} = \delta^2\,y,$$

$$i^3\,\psi''\,x = \frac{i^3}{1\,.\,2\,.\,3}\,.\,\frac{d^3\,.\,f}{d\,i^3} = \delta^3\,y,$$

. . . .

$$i^n\,\psi^{(n-1)}\,x = \frac{i^n}{1\,.\,2\,.\,3\,.\,.\,.n}\,.\,\frac{d^n\,.\,f}{d\,i^n} = \delta^n y.$$

10. A quantity y, or $f\,x$, may be supposed to change its value in various ways. It may alter in consequence of the variable x receiving an increment, the form of the function being the same; or by a change in the nature of the function by the introduction of a new variable, the former remaining constant; or generally, both the function and the variables which enter into it, may alter simultaneously, since the latter may become functions of a new independent variable when the former is changed.

As an illustration, let us suppose the equation

$$y = f\,x$$

to represent the equation to the curve $C\,E$, (fig. 2,) A being the origin of the co-ordinates and $M\,M'$ an element of the curve, $A\,P$, $A\,P'$ two consecutive values of x, and $P\,M$, $P'\,M'$ the corresponding values of y: the symbol d will represent the passage from the point M to M' in the same curve. But supposing the curve $C\,E$, by the operation of some cause, to be changed into the curve $C_1\,E_1$, the equation $y = f\,x$ will become changed into $y = f\,(x, i)$, which represents the equation to the corresponding new curve; the quantity i depending upon the cause or time or other circumstances which occasion the change of curve. Hence to a common abscissa $A\,P$, two ordinates $P\,M$, $P\,M_1$

correspond, whose difference, when i is infinitely small, is represented by δy; the letter δ indicating the transition of a point from one curve to another, and, when prefixed to a variable quantity, representing, like the letter d of the Differential Calculus, both an operation and a quantity. But when this variation is not restricted to the ordinate, or when the cause which alters the curve, changes likewise the abscissa, the point M may correspond to the point μ, in which case the abscissa will be subject to variation, and will become a function of the new variable i, as is generally the case in the disturbing actions of the planets, in which i is the disturbing force.

Let $AP = x$, $PM = y$; draw the ordinate μS, and MnN parallel to AP, and indicating for a moment, by δ and δ', the two preceding modes of variation, we shall have

$$\begin{array}{ll} dx = PP', & dy = M'N, \\ \delta x = 0, & \delta y = MM_1, \\ \delta' x = PS, & \delta' y = \mu n. \end{array}$$

11. In general, we may consider any function, and the variables which it contains, as also the differential coefficients, which are other functions of the same variables, subject to simultaneous variation; so that the values of the quantities x, y, z; x', y', z', &c.,

after receiving increments of variation *, will be $x + \delta x, y + \delta y, z + \delta z; x' + \delta x', y' + \delta y', z' + \delta z'$, &c. ; in the same manner as the values of the same quantities, after receiving differentials, are $x + dx$, $y + dy, z + dz ; x' + dx', y' + dy', z' + dz'$, &c.

Hence the differential coefficients, which are in-

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* $dx, \delta x$, &c., in physical inquiries, are magnitudes which, though infinitely small, have no less a real existence than finite quantities. When two straight lines meet another straight line, to take one of the simplest instances, the ratio of the points of intersection are the same as that of the cosecants of the angles with which they meet the straight line, which ratio may become indefinitely great. If one line make an angle of 30° with it, and the other be perpendicular, these points are to one another as 2 to 1 ; and though they are of no finite magnitude, they have manifestly the same existence as the straight lines in question. The ancient geometers only admitted finite magnitudes into their investigations ; and the principal advantages attained by the moderns, have arisen from conceiving *elements* or quantities infinitely small as subject to the same operations as finite magnitudes. In comparing finite quantities, we reject quantities infinitely less, even of the first order ; so in deducing finite values from the consideration of the ratios of infinitely small quantities of any order, which is the object of the Differential and Integral Calculus, we, with equal rigor, reject infinitely small quantities of a higher order, producing with the independent variable, ratios that vanish. In fact, from the generation of infinitely small quantities, it is manifest that no multiple of one of any order can be equal to one of the next inferior order. This constitutes the principle of homogeneity of infinitely small quantities.

dependent of the magnitude of the increments, and of the mode in which they are supposed to arise, will be common to both operations; and the rules for differentiating will apply to the Calculus of Variations, remembering to mark with the symbol δ, the characteristic principle of the Calculus.

If L be a function of $x, y, \dots . p, q, \dots .$ represented by

$$L = F\,(x, y \,\dots\dots\, p, q \dots\dots),$$

the complete differential of L will be

$$dL = \frac{dL}{dx}\,dx + \frac{dL}{dy}\,dy + \dots$$

$$+ \frac{dL}{dp}\,dp + \frac{dL}{dq}\,dq + \dots$$

and its variation will be represented by

$$\delta L = \frac{dL}{dx}\,\delta x + \frac{dL}{dy}\,\delta y + \dots$$

$$+ \frac{dL}{dp}\,\delta p + \frac{dL}{dq}\,\delta q + \dots$$

12. When the symbols d^n and δ are prefixed to the same quantity, it is indifferent as to the order in which they are placed, or in other words, which operation is first performed.

Since the variation of y is an infinitely small function of y, (9,) let us represent it by $i\,\phi y$, so that

$$\delta y = i\,\phi y,$$

and if $y_1 = y + d\,y$, we shall similarly have,

$$\begin{aligned} \delta y_1 &= i\,\phi y_1, \\ \therefore \delta y_1 - \delta y &= i\,\phi y_1 - i\,\phi y \\ &= d \cdot i\phi y; \end{aligned}$$

but $\delta y_1 - \delta y = \delta\, d\,y$, and $d \cdot i\phi y = d\,\delta y$; hence,

$$\delta dy = d\,\delta y.$$

This may be otherwise shewn by means of geometrical considerations.

Let $A\,P = x$, (fig. 3,) $P\,M = y$, then $P\,P' = dx$, $M'N = d\,y$, $M\,M_1 = \delta y$, $P'M' = y + d\,y$, $P\,M_1 = y + \delta y$; drawing $M'_1\,r$ parallel to the element $M\,M'$ and $M\,s$ parallel to $A\,X$, we have,

$$\begin{aligned} M'_1 r = M'_1 s - M'N &= d \cdot P\,M_1 - d \cdot P\,M \\ &= d\,(y + \delta y) - d\,y \\ &= d\,\delta y. \end{aligned}$$

But,

$$\begin{aligned} M'M'_1 &= \delta\,P'M' = \delta\,(y + d\,y), \\ \therefore M'_1 r &= \delta\,(y + dy) - \delta y \\ &= \delta\,d\,y; \end{aligned}$$

hence,

$$d\,\delta y = \delta\,d\,y.$$

When the function and the variable change simultaneously, the variation of y will consist of two parts; one, owing to the change of function, and the other part arising from the variation of x; representing this part by $\pi\,(i\,\phi\,x)$, and denoting, as before, for the sake of distinction, the total variation of y by $\delta' y$, we shall have,

$$\delta' y = i\,\phi\,y + \pi\,(i\,\phi\,x).$$

Similarly, if $y + dy = y_1$, we shall have,

$$\delta' y_1 = i\,\phi\,y_1 + \pi\,(i\,\phi\,x);$$

$$\begin{aligned} \therefore \delta' d\,y &= i\,\phi\,y_1 - i\,\phi\,y + \pi\,(i\,\phi\,x_1) - \pi\,(i\,\phi\,x) \\ &= d \cdot i\,\phi\,y + d \cdot \pi\,(i\,\phi\,x) \\ &= d\,\{(i\,\phi\,y + \pi\,(i\,\phi\,x)\} \\ &= d\,\delta' y. \end{aligned}$$

Otherwise:

Let $M M'$, (fig. 4,) be an element of the curve $A Z$, whose equation is $y = f x$, and $\mu\,\mu'$ the corresponding element after a variation: draw the lines $P M M_1$, $\mu\,S$, $P' M' M'_1$ and $\mu'\,S'$, perpendicular, and $M n$, $M_1 s$, $M'\,n'$, parallel to the axis $A X$.

The total variation of y, $n\,\mu$, consists of the two distinct parts $M M_1$ and $\mu\,s$; and since,

$$\begin{aligned} P M \text{ or } y &= f x, \\ P M_1 &= y + \delta y, \\ &= f x + i\,\psi\,x; \end{aligned}$$

let this be represented by the function $\varepsilon\,x$.

Also, $AS = x + \delta x$

$= x + i\phi x$,

which we shall denote by X;

also $AP' = x + dx$

$= x'$ (suppose)

$\therefore AS'$ will be denoted by X'.

Now, $S\mu = \varepsilon X$, $S'\mu' = \varepsilon X'$,

$\therefore \mu n$, or $\delta' y = \varepsilon X - fx$,

$\mu' n'$, or $\delta' y' = \varepsilon X' - fx'$;

hence, $\delta' y' - \delta' y$, or,

$$\delta' dy = \varepsilon X' - \varepsilon X - (fx' - fx),$$

$$= d(\varepsilon X - fx),$$

$$= d\,\delta' y, \text{ as before.}$$

Hence it follows that

$$d^2\delta y = d\,d\,\delta y = d\,\delta\,dy = \delta\,d\,dy = \delta\,d^2 y,$$

and generally,

$$d^n \delta y = \delta d^n y.$$

It will now be easy to extend this principle to any function U containing

$$x,\ dx,\ d^2x, \text{ \&c.},\ y,\ dy,\ d^2y, \text{ \&c.}$$

The variation of U in this case is found by substituting for x, dx, d^2x, &c., y, dy, d^2y, &c., the quantities

$$x + \delta x,\ dx + \delta dx,\ d^2x + \delta d^2x, \text{ \&c.}$$

$$y + \delta y,\ dy + \delta dy,\ d^2y + \delta d^2y, \text{ \&c.},$$

and subtracting from the result the value of U, limiting the expression to the first powers of the variations. But it will be more convenient if we make

$$dx = x',\ d^2x = x'',\ \&c.,\ dy = y',\ d^2y = y'',\ \&c.,$$

and consider U simply a function of

$$x,\ x',\ x'',\ \&c.,\ y,\ y',\ y'',\ \&c.$$

The differential of U being

$$dU = \frac{dU}{dx}dx + \frac{dU}{dx'}dx' + \frac{dU}{dx''}dx'' + \&c.$$

$$+ \frac{dU}{dy}dy + \frac{dU}{dy'}dy' + \frac{dU}{dy''}dy'' + \&c.;$$

the variation of U will be

$$\delta U = \frac{dU}{dx}\delta x + \frac{dU}{dx'}\delta x' + \frac{dU}{dx''}\delta x'' + \&c.$$

$$+ \frac{dU}{dy}\delta y + \frac{dU}{dy'}\delta y' + \frac{dU}{dy''}\delta y'' + \&c.$$

Taking the variation of the former and the differential of the latter, we shall have, respectively,

$$\delta dU = \frac{dU}{dx}\delta dx + \frac{dU}{dx'}\delta dx' + \&c.$$

$$+ \left(\frac{d^2U}{dx^2}\delta x + \frac{d^2U}{dx\,dx'}\delta x' + \&c.\right)dx$$

$$+ \left(\frac{d^2U}{dx\,dx'}\delta x + \frac{d^2U}{dx'^2}\delta x' + \&c.\right)dx'$$

$$+ \&c.$$

$$d\delta U = \frac{dU}{dx} d\delta x + \frac{dU}{dx'} d\delta x' + \&c.$$
$$+ \left(\frac{d^2 U}{dx^2} dx + \frac{d^2 U}{dx' dx} dx' + \&c. \right) \delta x$$
$$+ \left(\frac{d^2 U}{dx' dx} dx + \frac{d^2 U}{dx'^2} dx' + \&c. \right) \delta x'$$
$$+ \&c.$$

and since $\delta d x = d \delta x$, $\delta d x' = d \delta x'$, &c., we shall have,

$$\delta d U = d \delta U.$$

Hence we conclude, as before, that

$$\delta d^n U = d^n \delta U.$$

13. There is also a similar theorem with respect to the sign of integration.

Let us represent $\int U$ by U_1, then we shall have,

$$d U_1 = U, \text{and } \delta d U_1 = \delta U,$$

transposing the symbols δ and d, we have,

$$d \delta U_1 = \delta U,$$

integrating,

$$\delta U_1 = \int \delta U,$$

and replacing U_1, by its value $\int U$, we have,

$$\delta \int U = \int \delta U;$$

which shews that the variation of the integral of a function is equal to the integral of the variation of the function.

Whence we conclude generally that

$$\delta \int^n U = \int^n \delta U.$$

This theorem reduces the subject from the variation of an indefinite integral, to the variation of the function which is affected with the integral sign.

SECTION IV.

Variation of U. [illegible]

14. The preceding theorems constitute the fundamental principles of the Calculus of Variations. We shall now apply them to the determination of the variation of the integral $\int U$, in which U is a given function of $x, y \dots dx, dy \dots d^2x$, &c.; the variables $x, y \dots$ not being considered connected, but their dependence upon the independent variable being understood.

Since $U = F(x, y \dots dx, dy, \dots d^2x \dots)$,

$$\begin{aligned} dU &= M\,dx + N\,d^2x + P\,d^3x + Q\,d^4x + \&c. \\ &\quad + m\,dy + n\,d^2y + p\,d^3y + q\,d^4y + \&c. \\ &\quad \cdots\cdots\cdots\cdots\cdots\cdots \\ &= M\,dx + N\,d\,dx + P\,d\,d^2x + Q\,d\,d^3x + \&c. \\ &\quad + m\,dy + n\,d\,dy + p\,d\,d^2y + q\,d\,d^3y + \&c. \\ &\quad \cdots\cdots\cdots\cdots\cdots\cdots \end{aligned}$$

and, by the similarity of the effects d and δ,

$$\delta U = M\delta x + N\delta dx + P\delta d^2x + Q\delta d^3x + \&c.$$
$$+ m\delta y + n\delta dy + p\delta d^2y + q\delta d^3y + \&c.$$

. . . .

$$\therefore \int\delta U, \text{or } (13), \delta\int U = \int(M\delta x + N\delta dx + P\delta d^2x + \&c.)$$
$$+ \int(m\delta y + n\delta dy + p\delta d^2y + \&c.).$$

. . . .

Integrating by parts,

$$\int M\delta x = \int M\delta x;$$
$$\int N\delta dx = \int Nd\delta x = N\delta x - \int dN\delta x;$$
$$\int P\delta d^2x = \int Pd^2\delta x = Pd\delta x - \int dPd\delta x$$
$$= Pd\delta x - dP\delta x + \int d^2P\delta x;$$
$$\int Q\delta d^3x = \int Qd^3\delta x = Qd^2\delta x - \int dQd^2\delta x$$
$$= Qd^2\delta x - dQd\delta x + d^2Q\delta x - \int d^3Q\delta x,$$

by writing dQ for P in the expression for $\int P\delta d^2x$, &c.

In like manner,

$$\int m\delta y = \int m\delta y$$
$$\int n\delta dy = n\delta y - \int dn\delta y,$$
$$\int p\delta d^2y = pd\delta y - dp\delta y + \int d^2p\delta y,$$
$$\int q\delta d^3y = qd^2\delta y - dqd\delta y + d^2q\delta y - \int d^3q\delta y,$$

&c.

. . . .

Collecting the coefficients of δx, $d\delta x$, δy, &c., we have,

$$\begin{aligned}
\delta \int U = & (N - dP + d^2 Q - d^3 R + \&c.)\, \delta x \\
& + (P - dQ + d^2 R - \&c.)\, d\delta x \\
& + (Q - dR + \&c.)\, d^2 \delta x \\
& + (R - \&c.)\, d^3 \delta x \\
& + \&c. \\
& + (n - dp + d^2 q - d^3 r)\, \delta y \\
& + (p - dq + d^2 r - \&c.)\, d\delta y \\
& + (q - dr + \&c.)\, d^2 \delta y \\
& + (r - \&c.)\, d^3 \delta x \\
& + \&c. \\
& \ldots\ldots\ldots\ldots \\
& + \int (M - dN + d^2 P - \&c.)\, \delta x \\
& + \int (m - dn + d^2 p - \&c.)\, \delta y. \\
& \ldots\ldots\ldots\ldots
\end{aligned}$$

This equation, may, for brevity, be expressed as follows:

$$\begin{aligned}
\delta \int U = & \alpha \delta x + \beta d \delta x + \gamma d^2 \delta x + \&c. \\
& + \alpha_{,} \delta y + \beta_{,} d \delta y + \gamma_{,} d^2 \delta y + \&c. \\
& \ldots\ldots\ldots \\
& + \int (\chi \delta x + \psi \delta y \ldots.).
\end{aligned}$$

15. The indeterminate function $\int U$ may be susceptible of an infinite number of values between the same limits, owing to the indeterminate nature of

the functions x, y, &c. The question then may be to find what relations must subsist among these quantities, arising from the elimination of the independent variable, that the function $\int U$ may be a maximum or minimum between the given limits. And these being determined, the corresponding value of $\int U$ will be greater in the first case, and less in the second, than for any other relations whatever. To satisfy this condition it is necessary that the substitution of $x + \delta x, y + \delta y$, &c. ... $dx + \delta dx$, &c., and of $x - \delta x, y - \delta y$, &c. ... $dx - \delta dx$, &c., for x, y, &c. ... dx, &c., in $\int U$ should, by both substitutions, be diminished, or by both be increased; but this can only happen when the sum of the terms involving the first powers of $\delta x, \delta y$ ... is equal to 0. For the increment of $\int U$ will have the same sign as $\delta x, \delta y$ ... which are of the first dimension, and it is evidently impossible that this increment can be constantly negative or constantly positive unless the preceding sum vanishes.

Hence we must have

$$\begin{aligned}\delta \int U = {} & \alpha \, \delta x + \beta \, \delta y + \text{\&c.} \\ & + \alpha_{,} \, d \delta x + \beta_{,} \, d \delta y + \text{\&c.} \\ & \cdot \; \cdot \; \cdot \; \cdot \; \cdot \; \cdot \; \cdot \; \cdot \\ & + \int (\chi \, \delta x + \psi \, \delta y \ldots) = 0\end{aligned}$$

for the equation common to the maximum and minimum.

If we represent

$$\alpha\,\delta x + \beta\,\delta y \ldots + \alpha_{,}\,d\,\delta x + \beta_{,}\,d\,\delta y \ldots$$

by ϕ, and observe that as it results from integration, it must be taken between limits: denoting therefore by ϕ_0, the value of ϕ at the first limit, and by ϕ_1 its value at the second limit, we shall have

$$0 = \phi_1 - \phi_0 + \int \chi\,\delta x + \int \psi\,\delta y + \ldots .$$

Now this equation consists of two parts which have no relation to each other: $\phi_1 - \phi_0$ depending upon the values of $\alpha, \beta, \ldots x, y \ldots$ at the limits, and not upon their general values, and $\delta x, \delta y \ldots$ are variations depending upon the functions relative to the independent variable (14); and as no relations can be assumed that will in all cases subsist, we conclude that the preceding equation can be satisfied generally, only by making,

$$\left.\begin{array}{c} \phi_1 - \phi_0 = 0, \\ \chi = 0, \psi = 0, \text{\&c.} \end{array}\right\} \ldots (1).$$

Prob. To shew the use of these equations we shall, as a first example, apply them to determine the relation which must subsist between x and y, that the integral,

$$\int \surd\,(d x^2 + d y^2),$$

between given limits, may be a minimum.

Here we have

$$\begin{aligned} U &= F\,(d\,x,\ d\,y) \\ &= \surd\,(d\,x^2 + d\,y^2)\,; \end{aligned}$$

$$\therefore M=0, m=0, N=\frac{d\,x}{\surd\,d\,x^2+d\,y^2}, n=\frac{d\,y}{\surd\,d\,x^2+d\,y^2},$$

$$P = 0, p = 0,\ Q = 0, \&c.$$

Since the limits are fixed and independent of one another, $\phi_1 = 0$, $\phi_0 = 0$; and the equations $\chi = 0$, $\psi = 0$ become

$$d \cdot \frac{d\,x}{\surd\,(d\,x^2 + d\,y^2)} = 0,\quad d\,.\,\frac{d\,y}{\surd(d\,x^2+d\,y^2)} = 0.$$

Integrating

$$\frac{d\,x}{\surd\,(d\,x^2 + d\,y^2)} = c \text{ and } \frac{d\,y}{\surd\,(d\,x^2 + d\,y^2)} = c',$$

therefore

$$\frac{d\,y}{d\,x} = \frac{c'}{c} = C,$$

and integrating again we have

$$y = C\,x + C',$$

the relation sought; C and C' being two arbitrary constants.

Since $\surd\,(d\,x^2 + d\,y^2)$ represents the element of a line, the preceding example determines the shortest distance between two given points to be a straight line.

If the element be referred to three rectangular co-ordinates x, y, z, we shall have to make the expression

$$\int U = \int \sqrt{(d x^2 + d y^2 + d z^2)} \text{ a minimum.}$$

Here,

$$M = 0,\ m = 0,\ \mu = 0,\ N = \frac{d x}{d s},\ n = \frac{d y}{d s},\ \nu = \frac{d z}{d s}$$

$P = 0$, &c.,

and the equations,

$$\chi = 0,\ \psi = 0,\ \omega = 0,$$

give

$$d\,.\,\frac{d x}{d s} = 0,\ d\,.\,\frac{d y}{d s} = 0,\ d\,.\,\frac{d z}{d s} = 0\,;$$

therefore

$$\frac{d x}{d s} = c,\ \frac{d y}{d s} = c',\ \frac{d z}{d s} = c''\,;$$

hence

$$\frac{d z}{d x} = C,\ \frac{d z}{d y} = C',$$

and, integrating,

$$z = C x + a, \quad z = C' y + a';$$

which are equations to the projections of a straight line in space upon the planes $x z$, $y z$.

Or, since $\frac{dz}{dx}, \frac{dz}{dy}$ represent the trigonometrical tangents of the angles which the projections of the line upon the planes $x\,z$, $y\,z$ make with the axes of x and y, the equations

$$\frac{dz}{dx} = \text{const.}, \quad \frac{dz}{dy} = \text{const.},$$

shew that these projections are straight lines: hence the line required is straight.

To determine the arbitrary constants introduced by integration, we must make the straight line pass through the given points. Let x_0, y_0, z_0, be the coordinates of the first point, and x_1, y_1, z_1, those of the second; then we have the equations

$$z_0 = C x_0 + a, \quad z_1 = C x_1 + a,$$
$$z_0 = C' y_0 + a', \quad z_1 = C' y_1 + a';$$

to determine the four quantities, C, C', a, a'.

Prob. As a further illustration, let us determine the shortest distance between two given plane curves fixed in position.

By the preceding example the shortest distance is a straight line.

Let x_0, y_0 be the values of x, y at the first limit, and x_1, y_1 their values at the second; and the equation, $\varphi_1 - \varphi_0 = 0$, becomes

$$\frac{dx_1}{ds_1}\,\delta x_1 + \frac{dy_1}{ds_1}\,\delta y_1 - \frac{dx_0}{ds_0}\,\delta x_0 - \frac{dy_0}{ds_0}\,\delta y_0 = 0.$$

Now as the limits are not given, their loci being the given curves, the variations δx_0, δy_0, δx_1, δy_1 are not equal to 0, but manifestly have the relations established by the differential equations to the given curves.

Let these equations be represented by

$$\Phi(x, y) = 0, \qquad \Phi'(x, y) = 0;$$

from which we deduce

$$d y = m\, d x, \qquad d y = n\, d x:$$

hence we have

$$\delta y_0 = m_0\, \delta x_0, \qquad \delta y_1 = n_1\, \delta x_1,$$

and the preceding equation becomes

$$\left(\frac{d x_1}{d s_1} + n_1 \frac{d y_1}{d s_1}\right) \delta x_1 - \left(\frac{d x_0}{d s_0} + m_0 \frac{d y_0}{d s_0}\right) \delta x_0 = 0;$$

but x_1, x_0, and consequently δx_1, δx_0, are entirely independent of each other; hence this equation can be satisfied only by supposing

$$\left(\frac{d x_1}{d s_1} + n_1 \frac{d y_1}{d s_1}\right) \delta x_1 = 0, \left(\frac{d x_0}{d s_0} + m_0 \frac{d y_0}{d s_0}\right) \delta x_0 = 0,$$

from which we have

$$d x_1 + n_1\, d y_1 = 0, \qquad d x_0 + m_0\, d y_0 = 0,$$

whence we deduce

$$1 + n_1 \frac{d y_1}{d x_1} = 0, \quad 1 + m_0 \frac{d y_0}{d x_0} = 0.$$

Now $\frac{dy_1}{dx_1}$, $\frac{dy_0}{dx_0}$ express the trigonometrical tangent of the angle which the straight line required makes with the axis of x; and m_0, n_1 represent respectively the tangents of the angles which the given curves at the limits, make with the same axis; therefore, since m_0 and $\frac{dy_0}{dx_0}$ are the tangents of two angles, the tangent of the difference of these angles is expressed by $\frac{m_0 - \frac{dy_0}{dx_0}}{1 + m_0 \frac{dy_0}{dx_0}}$, which, since the denominator vanishes, is infinite; hence it follows, that the difference of the two angles is equal to a right angle. We therefore conclude that the straight line must cut the given curve at the first limit at right angles. And since the equation at the second limit is similar to that at the first, the same takes place at the second limit.

The preceding examples may serve to shew the use and mode of application of the equations $\phi_1 - \phi_0 = 0$, $\chi = 0$, $\psi = 0$, &c., relative to the general integral $\int U$.

16. But when the integral is of the form $\int V dx$, in which V is a given function of $x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}$, &c.,

x being the independent variable, and consequently $d\,x$ constant; its variation will take a different and generally a more convenient form for the solution of problems.

Let K be a function of x and other quantities depending upon x; then we shall represent by K', K'', K''', &c., the differential coefficients of K taken with respect to x, and the other quantities depending upon x; so that

$$\frac{d\,(K)}{d\,x} = K', \frac{d\,(K')}{d\,x} = K'', \frac{d(K'')}{dx} = K''', \text{\&c.}$$

Also the two limits of any integral with respect to x, being x_0 and x_1, the values of any quantity H, corresponding to the same limits, will be denoted by H_0, at the limit x_0, and H_1 at the limit x_1.

Again, since y is a function of x, we shall have from the above notation,

$$\frac{d\,y}{d\,x} = y', \frac{d\,y'}{d\,x} = y'', \frac{d\,y''}{d\,x} = y''', \text{\&c.},$$

and the quantity V will be a given function of x, y, y', y'', &c. Now x_0 and x_1 being two constant quantities, or quantities independent of x, we proceed to determine the variation of U in the equation

$$U = \int_{x_0}^{x_1} V\,d\,x.$$

Since

$$V = F(x, y, y', y'', \&c.),$$
$$dV = M\,dx + N\,dy + P\,dy' + Q\,dy'' + \&c.,$$

in which

$$M = \frac{dV}{dx},\ N = \frac{dV}{dy},\ P = \frac{dV}{dy'},\ Q = \frac{dV}{dy''},\ \&c.,$$

and

$$\delta V = M\,\delta x + N\,\delta y + P\,\delta y' + Q\,\delta y'' + \&c.$$

Now, by (13), we have , p. 31.

$$\delta \int_{x_0}^{x_1} V dx = \int_{x_0}^{x_1} \delta . V dx = \int_{x_0}^{x_1} V d\delta x + \int_{x_0}^{x_1} \delta V dx$$

$$= V_1\,\delta x_1 - V_0\,\delta x_0 - \int_{x_0}^{x_1} dV\,\delta x + \int_{x_0}^{x_1} \delta V\,dx$$

$$= V_1\,\delta x_1 - V_0\,\delta x_0 + \int_{x_0}^{x_1} (dx\,\delta V - dV\,\delta x) \ldots (1).$$

In the expression under the sign $\int$, substitute the values of dV and δV given above, and we have

$$dx\,\delta V - \delta x\,dV = N(\delta y - y'\delta x)\,dx + P(\delta y' - y''\delta x)\,dx$$
$$+ Q(\delta y'' - y'''\delta x)\,dx + \&c. \ldots\ldots\ldots\ (2);$$

but,

$$\delta y' = \delta \frac{dy}{dx} = \frac{dx\,\delta dy - dy\,\delta dx}{dx^2} = \frac{d\delta y - y' d\delta x}{dx};$$

similarly,

$$\delta y'' = \frac{d\,\delta y' - y'' d\,\delta x}{d x},\ \delta y''' = \frac{d\,\delta y'' - y''' d\,\delta x}{d x}, \text{ \&c.}$$

therefore, $\left(P\, dx\left(\delta y' - \frac{dy'}{dx}\delta x\right)\right.$, or;

$$\begin{aligned} d x\,\delta y' - d y'\,\delta x &= \overline{d\,\delta y - y' d\,\delta x} - d y'\,\delta x \\ &= d\,\delta y - d\,.\,y'\,\delta x \\ &= d\,(\delta y - y'\,\delta x). \end{aligned}$$

Changing y' into y'', and y into y', we have

$$d x\,\delta y'' - d y''\,\delta x = d\,(\delta y' - y''\,\delta x);$$

in the same way we obtain

$$d x\,\delta y''' - d y'''\,\delta x = d\,(\delta y'' - y'''\,\delta x),$$

&c.

For brevity, let $\delta y - y'\,\delta x = \omega$,

then,

$$\begin{aligned} \delta y' - y''\,\delta x &= \omega', \\ \delta y'' - y'''\,\delta x &= \omega'', \end{aligned}$$

&c.

Substituting these values in equ. (2) and affecting each term with the integral sign between the limits of $x = x_0$ and $x = x_1$, we have

$$\int_{x_0}^{x_1} (d x\,\delta V - d V\,\delta x) = \int_{x_0}^{x_1} N \omega\, d x + \int_{x_0}^{x_1} P \omega' d x$$

$$+ \int_{x_0}^{x_1} Q\,\omega'' d x + \int_{x_0}^{x_1} R\,\omega''' d x + \text{\&c.}$$

Integrating by parts,

$$\int_{x_0}^{x_1} P\,\omega'\,dx = P_1\,\omega_1 - P_0\,\omega_0 - \int_{x_0}^{x_1} P'\,\omega\,dx,$$

$$\int_{x_0}^{x_1} Q\,\omega''\,dx = Q_1\,\omega'_1 - Q_0\,\omega'_0 - \int_{x_0}^{x_1} Q'\,\omega'\,dx$$

$$= Q_1\,\omega'_1 - Q_0\,\omega'_0 - Q'_1\,\omega_1 + Q'_0\,\omega_0 + \int_{x_0}^{x_1} Q''\,\omega\,dx,$$

$$\int_{x_0}^{x_0} R\,\omega'''\,dx = R_1\,\omega''_1 - R_0\,\omega''_0 - R'_1\,\omega'_1 + R'_0\,\omega'_0$$

$$+ R''_1\,\omega_1 - R''_0\,\omega_0 - \int_{x_0}^{x_1} R'''\,\omega\,dx,$$

&c.

Collecting these terms and substituting them in p. 44 equation (1), we obtain, for the variation sought,

$$\delta U \text{ or } \delta\int_{x_0}^{x_1} V\,dx$$

$$\begin{aligned} = V_1\,\delta x_1 - V_0\,\delta x_0 &+ (P_1 - Q'_1 + R''_1 - \&c.)\,\omega_1 \\ &- (P_0 - Q'_0 + R''_0 - \&c.)\,\omega_0 \\ &+ (Q_1 - R'_1 + \&c.)\,\omega'_1 \\ &- (Q_0 - R'_0 + \&c.)\,\omega_0 \\ &+ \&c. \end{aligned}$$

$$+ \int_{x_0}^{x_1} (N - P' + Q'' - R''' + \&c.)\,\omega\,dx,$$

in which we must remember that P being a function of x, y, y', y'', &c.,

$$P' \text{ or } \frac{d(P)}{dx} = \frac{dP}{dx} + \frac{dP}{dy} \cdot \frac{dy}{dx} + \frac{dP}{dy'} \cdot \frac{dy'}{dx} + \frac{dP}{dy''} \cdot \frac{dy''}{dx} + \&c.$$

a similar meaning attaches to Q', R', &c., Q'', &c. The above expression for the variation of U consists of two distinct parts, and may be written as follows:

$$\delta U = \Gamma + \int_{x_0}^{x_1} \mathrm{H}\,\omega\,dx \;\ldots\ldots\; (A),$$

making, for brevity,

$$\mathrm{H} = N - P' + Q'' - R''' + \&c.,$$

$$\begin{aligned}\Gamma = {} & V_1\,\delta x_1 - V_0\,\delta x_0 \\ & + (P_1 - Q'_1 + R''_1 - \&c.)\,\omega_1 - (P_0 - Q'_0 + R''_0 - \&c.)\,\omega_0 \\ & + (Q - R' + \&c.)\,\omega'_1 - (Q - R'_0 + \&c.)\,\omega'_0 \\ & + (R_1 - \&c.)\,\omega''_1 - (R_0 - \&c.)\,\omega''_0 \\ & + \&c.\end{aligned}$$

If the quantity ω, under the sign $\int$, be replaced by its value, $\delta y - y'\,\delta x$, and the quantities,

$$\omega_0,\ \omega_1,\ \omega'_0,\ \&c.,$$

in the term Γ, by their values,

$$\delta y_0 - y'_0\,\delta x,\ \delta y_1 - y'_1\,\delta x_1,\ \delta y'_0 - y''_0\,\delta x,\ \&c.,$$

the variation of U will be expressed implicitly by the variations of x and y, and the variations of the extreme limits of $x, y, y', y''\ldots$, continued to a differential coefficient immediately inferior to the highest contained in V.

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17. When the quantities, x_0, y_0, y'_0, &c., x_1, y_1, y'_1, &c., or any of them, are contained in V, it is necessary to add to the term Γ of the equation (A), other terms arising from the variations of these quantities. By reviewing the processes followed with regard to x, y, y', y'', &c., in the preceding investigation, it is manifest that these terms will be, respectively,

$$\delta x_0 \int_{x_0}^{x_1} \frac{dV}{dx_0} dx,\ \delta y_0 \int_{x_0}^{x_1} \frac{dV}{dy_0} dx,\ \delta y'_0 \int_{x^0}^{x_1} \frac{dV}{dy'_0} dx, \text{ \&c.,}$$

$$\delta x_1 \int_{x_0}^{x_1} \frac{dV}{dx_1} dx, \text{ \&c.}$$

so that, in this case, the quantity Γ will no longer be delivered from the sign $\int$; but since we confine ourselves to the first powers of the variations (9), it will still be a linear function of δx_0, δy_0, &c., δx_1, δy_1, &c.

18. If the variations δx and δy have the same ratio as the differentials dx and dy for all values of x between the given limits, in which case, $\delta y - y' \delta x$ or $\omega = 0$, the variation of U will be reduced to

$$V_1 \delta x_1 - V_0 \delta x_0,$$

which is, in fact, what ought to happen, since, in this case,

$$\delta . \int_{x_0}^{x_1} V\,dx = V_1\,\delta x_1 - V_0\,\delta x_0.$$

Let x and y be the co-ordinates of any point in a curve, and suppose that the variations have the same ratio as the differentials in the whole extent of the curve, this will not be changed, and the integral will only be augmented by $V_1\,\delta x_1$ at the limit x_1, and diminished by $V_0\,\delta x_0$ at the limit x_0.

19. Suppose now we have to determine the function y in terms of x and of quantities depending upon the limits of U, so that this integral may be a maximum or minimum between those limits. Let α be an infinitely small quantity, and suppose that in the expression for the total variation of U, (which arises from the substitution of $x + \delta x$ and $y + \delta y$ for x and y and subtracting its primitive value) we replace δx and δy by $\alpha\,\delta x$ and $\alpha\,\delta y$, the total increment of U will be expressed in a series of ascending powers of α, the first term of which will be $\alpha\,\delta U'$ as is evident from art. (15.) And since this term changes its sign with α, it may be shewn, precisely as in the maxima and minima of the Differential Calculus, that

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$$\delta U = 0;$$

which is the equation common to the maximum and minimum.

The distinction between the maximum and minimum will be indicated, as in the Differential Calculus, by the sign which the second term in the above series takes; being in the former case, negative, and in the latter, positive. In other words, the maximum or minimum takes place, according as the value of $\int \delta^2 V\, d x$, in the whole extent between the same limits as U, is negative or positive; $\delta^2 V$, representing that part of the variation of V which contains the terms of the second dimension with respect to δy, $\delta y'$, $\delta y''$, &c. Let us suppose that in U, or $\int V\, d x$, the function V contains only x, y, and y', the complete development of the variation of U, as far as terms of the second order, will be obtained by putting for V, the expression

$$V + \frac{d V}{d y} \delta y + \frac{d V}{d y'} \delta y'$$

$$+ \tfrac{1}{2} \left(\frac{d^2 V}{d y^2} \delta y^2 + 2 \frac{d^2 V}{d y\, d y'} \delta y\, \delta y' + \frac{d^2 V}{d y'^2} \delta y'^2 \right)$$

in the integral $\int V\, d x$.

The terms of the first order, $\frac{d V}{d y} \delta y + \frac{d V}{d y'} \delta y'$, multiplied by $d x$, and integrated, will give the result which is contained in the equation (A). In

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the case of maximum or minimum, this expression $= 0$; so that the variation of U is reduced to

$$\tfrac{1}{2}\int_{x_0}^{x_1}\left(\frac{d^2 V}{dy^2}\delta y^2 + 2\frac{d^2 V}{dy\,dy'}\delta y\,\delta y' + \frac{d^2 V}{dy'^2}\delta y'^2\right)dx;$$

and this term, containing factors of the second dimension with respect to δy, $\delta y'$, will be independent of the sign of δy, $\delta y'$; hence, it is manifest, that according as this quantity is negative or positive for all values of x between x_0 and x_1, the integral U will be a maximum or minimum.

In the first example, art. (25.), p. 64

$$V = \sqrt{1 + y'^2},$$

the above expression is reduced to its last term, and we have

$$\int \frac{d^2 V}{dy'^2}\,dx = \int \frac{dx}{(1 + y'^2)^{\frac{3}{2}}},$$

a quantity essentially positive; hence the integral,

$$U = \int dx\sqrt{1 + y'^2}$$

belongs to a minimum.

20. The equation $\delta U = 0$, corresponds to a maximum and minimum; and since this condition exists in both cases, whatever be the values of δx and δy, which are entirely arbitrary, and ω being, in

consequence, an arbitrary function of x, it is necessary that the integral contained in the equation (A), p. and the part freed from the sign $\int$, should separately be $= 0$. For the same reason, this integral, cannot, generally, vanish, unless $\mathrm{H} = 0$. We have then the two conditions,

$$\Gamma = 0, \qquad \mathrm{H} = 0,$$

the first of which is called the equation of limits, and the second, or

$$N - P' + Q'' - R''' + \&c. = 0,$$

contains the relation which must subsist between x and y, in order that the proposed integral may be a maximum or minimum: it will make known, by integration, the function y, in terms of x, and a certain number of arbitrary constants; these constants, and the two limits x_0 and x_1, will be determined from equations deduced from the equation $\Gamma = 0$, and the particular conditions to which the limits of U are subjected.

21. If V be a differential function of the order n, that is, if it contain $y^{(n)}$, the equation $\mathrm{H} = 0$, will be of the order $2n$, (16.), and its complete integral, which we shall represent by $\mathrm{X} = 0$, will contain $2n$ arbitrary constants, $c, c', c'' \ldots c^{(2n-1)}$. Now if

the limits of U are not subject to any given conditions, the variations,

$$\delta x_0, \delta y_0, \delta y'_0, \ldots \delta y_0^{(n-1)}, \delta x_1, \delta y_1, \delta y'_1, \ldots \delta y_1^{(n-1)},$$

of which the term Γ is a linear function, are entirely independent of one another; their coefficients must therefore be separately $= 0$, which will furnish a number of equations equal to $2n + 2$, and which, joined to the $2n$ equations,

$$X_0 = 0, X'_0 = 0 \ldots X_0^{(n-1)} = 0, X_1 = 0, X'_1 = 0 \ldots X_1^{(n-1)} = 0,$$

will give $4n + 2$ equations to determine the $2n$ constants, $c, c', c'' \ldots c^{(2n-1)}$, and the $2n + 2$ quantities $x_0, y_0, y', \ldots y_0^{(n-1)}, x_1, y_1, y'_1, \ldots y_1^{(n-1)}$.

But if the limits of U be connected by the equations,

$$A = 0, \qquad B = 0;$$

we shall have,

$$\frac{dA}{dx_0}\delta x_0 + \frac{dA}{dy_0}\delta y_0 + \&c. + \frac{dA}{dx_1}\delta x_1 + \frac{dA}{dy_1}\delta y_1 + \&c. = 0,$$

$$\frac{dB}{dx_0}\delta x_0 + \frac{dB}{dy_0}\delta y_0 + \&c. + \frac{dB}{dx_1}\delta x_1 + \frac{dB}{dy_1}\delta y_1 + \&c. = 0;$$

and by means of these equations, we may eliminate as many of the quantities, δx_0, δy_0, &c., contained in $\Gamma = 0$; the coefficients of the remaining variations equated to 0, joined to the equations, $A = 0$,

$B = 0$, and to the equations $X_0 = 0$, $X_1 = 0$, &c., will still furnish $4n + 2$ equations, as in the preceding case, to determine as many unknown constant quantities.

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22. The method upon which the investigation in (16.) is founded, may readily be extended to the case in which the integral V contains several functions. Let y and z be two unknown functions of the independent variable x, also suppose V to be a given function of x, y, y', y'', &c., z, z', z'', &c., and the integral to be of the same form as before, viz.,

$$U = \int_{x_0}^{x_1} V\, d\,x.$$

If we make

$$\frac{d\,V}{d\,x} = M, \frac{d\,V}{d\,y} = N, \frac{d\,V}{d\,y'} = P, \frac{d\,V}{d\,y''} = Q, \text{ \&c.}$$

$$\frac{d\,V}{d\,z} = n, \frac{d\,V}{d\,z'} = p, \frac{d\,V}{d\,z''} = q, \text{ \&c.}$$

we shall have

$$\begin{aligned} \delta\,V = M\,\delta\,x + N\,\delta\,y + P\,\delta\,y' + Q\,\delta\,y'' + \text{\&c.} \\ + n\,\delta\,z + p\,\delta\,z' + q\,\delta\,z'' + \text{\&c.} \end{aligned}$$

Now if we suppose

$$\begin{aligned}
\mathrm{H} &= N - P' + Q'' - \&c. \\
\mathrm{K} &= n - p' + q'' - \&c. \\
\Gamma &= V_1\,\delta x_1 + (P_1 - Q'_1 + \&c.)\,(\delta y_1 - y'_1\,\delta x_1) \\
&\qquad + (p_1 - q'_1 + \&c.)\,(\delta z_1 - z'_1\,\delta x_1) \\
&\qquad + (Q' - \&c.)\,(\delta y'_1 - y''_1\,\delta x_1) \\
&\qquad + (q' - \&c.)\,(\delta z'_1 - z''_1\,\delta x_1) \\
&\qquad + \&c. \\
&- V_0\,\delta x_0 - (P_0 - Q'_0 + \&c.)\,(\delta y_0 - y'_0\,\delta x_0) \\
&\qquad - (p_0 - q'_0 + \&c.)\,(\delta z_0 - z'_0\,\delta x) \\
&\qquad - (Q_0 - \&c.)\,(\delta y'_0 - y''_0\,\delta x_0) \\
&\qquad - (q_0 - \&c.)\,(\delta z'_0 - z''_0\,\delta x_0) \\
&\qquad - \&c.,
\end{aligned}$$

we shall find

$$\delta U = \Gamma + \int_{x_0}^{x_1} \{\mathrm{H}(\delta y - y'\delta x) + \mathrm{K}(\delta z - z'\delta x)\} dx \ldots (1)$$

for the increment of U, corresponding to the increments, δx, δy, δz, of x, y, z.

If V should contain besides x, y, y', &c., z, z', &c., the quantities x_0, x_1, y_0, y_1, z_0, &c., we must add to the preceding formula, the part

$$\delta x_0 \int_{x_0}^{x_1} \frac{dV}{dx_0} dx + \delta x_1 \int_{x_0}^{x_1} \frac{dV}{dx_1} dx + \delta y_0 \int_{x_0}^{x_1} \frac{dV}{dy_0} dx + \&c.$$

which arises from the variations of x_0, x_1, y_0, &c., as in (17). p. 48

The equation $\delta U = 0$ which takes place when U is a maximum or minimum, may be separated into the two equations,

$$\mathrm{H}(\delta y - y' \delta x) + \mathrm{K}(\delta z - z' \delta x) = 0,$$
$$\Gamma = 0;$$

the first of which exists in the whole extent of the integral; and the latter, as the equation of limits.

The former equation may be put under this form :

$$\mathrm{H}\,\delta y + \mathrm{K}\,\delta z - (\mathrm{H}y' + \mathrm{K}z')\,\delta x = 0;$$

which, as we shall see presently, furnishes only the two equations $\mathrm{H} = 0$, $\mathrm{K} = 0$, corresponding to the variations of y and z; the third, depending upon the variation of x, being contained in the other two. Whence it follows, that we may dispense with attributing any variation to x, whose differential is supposed constant in the function V, since the equations necessary for the solution of the general problem, result solely from the variations of the other variables.

Let L be a given function of x, y, z, and suppose the unknown quantities of y and z to be connected by the equation $L = 0$,
we shall have

$$\frac{dL}{dx}\delta x + \frac{dL}{dy}\delta y + \frac{dL}{dz}\delta z = 0.$$

But by differentiating the equation $L = 0$ with respect to x, we also have

$$\frac{dL}{dx} + \frac{dL}{dy} y' + \frac{dL}{dz} z' = 0,$$

by means of which the preceding equation becomes

$$\frac{dL}{dy}(\delta y - y' \delta x) + \frac{dL}{dz}(\delta z - z' \delta x) = 0,$$

which enables us to eliminate one of the quantities $\delta y - y' \delta x$, $\delta z - z' \delta x$ from the equation,

$$\mathrm{H}(\delta y - y' \delta x) + \mathrm{K}(\delta z - z' \delta x) = 0;$$

then, equating the coefficient of the other to 0, we determine one of the unknown quantities, y or z, in terms of x, and a certain number of arbitrary constants, corresponding to the order of the equation; and the equation $L = 0$, will determine the other.

Instead of eliminating one of the quantities, $\delta y - y' \delta x$, $\delta z - z' \delta x$, we may add the equations $\delta y - y' \delta x = 0$ and $dz - z' \delta x = 0$ after multiplying one of them by an indeterminate factor λ [*], and then equate to zero, the coefficients of the two quantities $\delta y - y' \delta x$ and $\delta z - z' \delta x$; in this manner we shall have

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[*] Vid. *Method of Multipliers*, Mécanique Analytique, Tom. I. p. 74.

$$H + \lambda \frac{dL}{dy} = 0, \; K + \lambda \frac{dL}{dz} = 0;$$

and these two equations, joined to $L = 0$, will make known the values of the quantities y, z, and λ. The arbitrary constants introduced by integration are determined by means of equations deduced from the condition $\Gamma = 0$, combined with $\delta L_0 = 0$ and $\delta L_1 = 0$, and from other conditions to which the limits of the integral may be subjected.

If the two unknown quantities y and z are independent, δy and δz are totally independent also; hence we must have

$$H = 0, \qquad K = 0.$$

There will now be no difficulty in extending the solution to the case in which V is a given function of three, or a greater number of unknown quantities, independent, or connected with each other by one or several given equations.

It may be remarked that when we make all the variables in the function V subject to variation, we only deduce as many equations $H = 0$, $K = 0$, &c., as there are variables less one, which is precisely what ought to result; since it is not the individual value of each of the variables that we require, as in the ordinary questions of maxima and minima, but the indefinite relations existing among these variables, by which they become functions of one another; and they may, consequently, be represented either by plane curves, or by curves of double curvature.

23. The equation

$$H = 0, \text{ or } N - P' + Q'' - R''' + \&c. = 0,$$

[illegible] — 47

that is,

$$N - \frac{d\,(P)}{d\,x} + \frac{d^2\,(Q)}{d\,x^2} - \frac{d^3\,(R)}{d\,x^3} + \&c. = 0,$$

may frequently be rendered more easy of application by previous integration.

1st. Suppose the function V contains only y and y', the equation $H = 0$ is reduced to

$$\begin{aligned} N - P' &= 0\,; \\ \therefore V' &= N\,y' + P\,y'', \\ &= P'\,y' + P\,y''\,; \end{aligned}$$

$\frac{dV}{dx} =$ [illegible]

integrating,

$$V = P\,y' + C, \cdots\cdots (\alpha)$$

C being an arbitrary constant quantity.

If V contains x, y and y', we shall have,

$$V = \int M\,d\,x + P\,y' + C. \cdots (\beta)$$

2d. Suppose V involve only y' and y'', the equation $H = 0$, then becomes,

$$P' = Q'',$$

integrating,

$$P = Q' + C;$$

$$\therefore V' = Py'' + Qy''',$$
$$= Q'y'' + Cy'' + Qy''';$$

and, integrating again we have

$$V = Qy'' + Cy' + C'; \text{ ---- } (\gamma)$$

C and C' being arbitrary constants.

If x also enters into V we shall have

$$V = \int M\,dx + Qy'' + Cy' + C'. \text{ --- } (\delta)$$

3d. Let V contain y, y' and y''; then,

$$N - P' + Q'' = 0,$$

or,

$$Ny' = P'y' - Q''y';$$

$$\therefore V' = P'y' - Q''y' + Py'' + Qy'''$$
$$= P'y' + Py'' - y'Q'' + Qy''';$$

integrating,

$$V = Py' - Q'y' + Qy'' + C. \text{ ---- } (\varepsilon)$$

4th. Suppose y' and y''' only to enter; then,

$$P' = -R'''$$

integrating,

R'' $$P = -R + C,$$

whence,

$$\begin{aligned} P y'' &= -R'' y'' + C y'', \\ \therefore V_{\wedge} &= P y'' + R y^{\text{IV}} \\ &= -R'' y'' + C y'' + R y^{\text{IV}}. \end{aligned}$$

Integrating, observing that

$$y'' R'' = \frac{d\,(y'' R')}{d\,x} - R' y''',$$

we have

$$V = -R' y'' + R y''' + C y' + C' \cdots\cdots (\zeta)$$

5th. Let y and y''' only enter into V; then,

$$N - R''' = 0,$$

or,

$$N y' = R''' y'.$$

$$V' = N y' + R y^{\text{IV}} = R''' y' + R y^{\text{IV}}$$

$$\therefore V' = \frac{d\,(R'' y')}{d\,x} - R'' y'' + R y^{\text{IV}}$$

$$= \frac{d\,(R'' y')}{d\,x} - \frac{d\,(R' y'')}{d\,x} + R' y''' + R y^{\text{IV}}.$$

Integrating,

$$V = R'' y' - R' y'' + R y''' + C. \cdots\cdots (\eta)$$

And similarly in other cases.

24. The quantity V being a given function of x, y, y', y'', &c., if $V\,dx$ is an exact differential without establishing any determinate relation between x and y, the definite integral U will be a function of the quantities x_0, y_0, y'_0, &c., x_1, y_1, y'_1, &c., relative to the two limits which contain no sign of integration. The variation δU given by the equation (A) will thus be reduced to the part Γ; whence the factor H comprised within the integral sign must become identically equal to 0. Thus the same equation $H = 0$, which determines the value of y relative to the maximum or minimum of U when $V\,dx$ is not an exact differential, ought to be identical when $V\,dx$ is an exact differential. This remark was originally made by Euler, who first expressed by an equation the condition necessary to the integrability of a differential formula.

If we multiply the equation

$$N - \frac{dP}{dx} + \frac{d^2 Q}{dx^2} - \frac{d^3 R}{dx^3} + \&c. = 0$$

by dx and integrate each term, we shall have

$$\int N\,dx - P + \frac{dQ}{dx} - \frac{d^2 R}{dx^2} + \&c. = C \ldots . (1).$$

$$\therefore \int N\,dx = C + P - \frac{dQ}{dx} + \frac{d^2 R}{dx^2} - \&c.$$

hence $N\,d\,x$ is integrable, since the second side contains only integrated functions.

Multiplying the equation (1) by $d\,x$ and integrating, we have

$$\int(\int N dx - P)\,d\,x + Q - \frac{d\,R}{d\,x} + \text{\&c.} = C\,x + C',$$

whence it follows that $\int(\int N\,d\,x - P)\,d\,x$ is also integrable. In the same manner we shall find that

$$\int\left\{\int(\int N\,d\,x - P)\,d\,x + Q\right\}d\,x,$$

$$\int\left\{\int\{\int(\int N\,d\,x - P)\,d\,x + Q\}\,d\,x - R\right\}d\,x,$$

&c.

are complete integrals.

SECTION V.

25. The preceding Section contains all the investigations which are necessary to the complete solution of problems involving the single consideration of a maximum or minimum. We shall now proceed to illustrate them by application to examples.

EXAMPLE I.

It is required to find the shortest distance between two given points.

Here we have

$$U = \int_{x_0}^{x_1} V\,dx = \int_{x_0}^{x_1} \sqrt{1+y'^2}\,dx;$$

$\therefore V = \sqrt{1+y'^2}$, into which y' only enters.

The equation $\Gamma = 0$ has no existence, and the equation $H = 0$ becomes, p. 52

$$P' = 0 \quad \therefore P = C;$$

$$\text{but } P = \frac{dV}{dy'} = \frac{y'}{\sqrt{1+y'^2}},$$

$$\therefore \frac{y'}{\sqrt{1+y'^2}} = C, \text{ or } y' = \frac{C}{\sqrt{1 \text{[illegible]} C^2}};$$

integrating, we have

$$y = \frac{C}{\sqrt{1 \text{[illegible]} C^2}}\, x + C',$$

the general equation to a straight line.

If the arbitrary constants C and C' are to be determined from the conditions $X_0 = 0$, $X_1 = 0$, (21), p 52 so that the straight line may pass through the given points (x_0, y_0), (x_1, y_1), we have

$$C = \frac{y_1 - y_0}{\sqrt{\{(x_1 - x_0)^2 + (y_1 - y_0)^2\}}}, \quad C' = \frac{x_1 y_0 - x_0 y_1}{x_1 - x_0};$$

and the preceding equation will become

$$y = \frac{y_1 - y_0}{x_1 - x_0}\, x + \frac{x_1 y_0 - x_0 y_1}{x_1 - x_0}.$$

EXAMPLE II.

Required the relation of x and y, so that the expression $\int (a x - y^2) y \, d x$ may be a maximum or minimum.

Here $U = \int (a x - y^2) y \, d x$, and $V = (a x - y^2) y$; hence the equation $H = 0$ becomes

$$N = 0,$$

$$\text{or,} \quad a x - 3 y^2 = 0,$$

which expresses the relation sought.

EXAMPLE III.

Let it be required to determine the shortest line that can be drawn from one given curve to another given curve.

a. The line required being straight, we have only to determine its position.

For this purpose we have the equation, $\Gamma = 0$, or

$$V_1 \delta x_1 - V_0 \delta x_0 + P_1 \omega_1 - P_0 \omega_0 = 0;$$

substituting for ω_1, ω_0 their values

$$\delta y_1 - y'_1 \delta x_1, \ \delta y_0 - y'_0 \delta x_0,$$

$$P_1 \delta y_1 + (V_1 - P_1 y'_1) \delta x_1 - P_0 \delta y_0 - (V_0 - P_0 y'_0) \delta x_0 = 0:$$

we have also the equations $A = 0$, $B = 0$ (21), p.5

which, in this case, are the equations to the two given curves; these determine the relations of the variations, δx_0, δy_0 and δx_1, δy_1.

In fact, let CE, GH, (fig. 5.), be the two given curves to which the extremities of a line, MR, which undergoes variation, are restricted; suppose $M'R'$ be consecutive in position to MR, and draw MP, $M'N$, RQ, $R'S$ perpendicular, and MN, RS parallel, to AX: then MM', RR' being elements of the curves CE, GH, let $AP = x_0$, $MP = y_0$, $AQ = x_1$, $RQ = y_1$; we shall have

$$dx_0 = MN = \delta x_0,\ dy_0 = M'N = \delta y_0,$$
$$dx_1 = RS = \delta x_1,\ dy_1 = R'S = \delta y_1.$$

Suppose $dy = m\,dx$, $dy = n\,dx$ to represent the equations deducible from $dA = 0$, $dB = 0$, then for the variations at the limits we have

$$\delta y_0 = m_0\,\delta x_0,\ \delta y_1 = n_1\,\delta x_1.$$

Also, (Example 1),

$$y'_0 = \frac{C}{\sqrt{1 + C^2}} = c,\ y'_1 = c\,;$$

$$V_0 - P_0 y'_0 = \sqrt{1 + y'^2_0} - \frac{y'^2_0}{\sqrt{1 + y'^2_0}} = \frac{1}{\sqrt{1 + y'^2_0}} = \frac{1}{\sqrt{1 + c^2}},$$

$$V_1 - P_1 y'_1 = \frac{1}{\sqrt{1 + c^2}},\ \text{and } P_0 = \frac{y'_0}{\sqrt{1 + y'^2_0}} = P_1\,;$$

hence, the equation of limits becomes

$$\left\{\frac{n_1 c}{\sqrt{1+c^2}}+\frac{1}{\sqrt{1+c^2}}\right\}\delta x_1 - \left\{\frac{m_0 c}{\sqrt{1+c^2}}+\frac{1}{\sqrt{1+c^2}}\right\}\delta x_0=0;$$

and since no relation is supposed to subsist between the given curves, none subsists between

$$\delta x_0 \text{ and } \delta x_1;$$

we must therefore assume each coefficient $= 0$;

$$\therefore 1 + n_1 c = 0, \quad 1 + m_0 c = 0,$$

or,

$$1 + \frac{d\,y_1}{d\,x_1} y'_1 = 0, \quad 1 + \frac{d\,y_0}{d\,x_0} y'_0 = 0;$$

whence it appears that the straight line must cut both the given curves at right angles.

To find the co-ordinates of the points in which the straight line meets the given curves, we have the equations,

$$A_0 = 0, \quad B_1 = 0,$$

$$1 + m_0 c = 0, \quad 1 + n_1 c = 0.$$

For instance, if the two given curves be parabolas, whose axes are at right angles to each other, and whose vertices are at the distances a and b from the origin of the rectangular axes, which we may suppose to coincide with the axes of the parabolas; the equations,

$$A = 0, \qquad B = 0,$$

will be replaced by

$$y^2 - p(x-a) = 0, \; x^2 - p'(y-b) = 0;$$

from which we deduce

$$\frac{dy}{dx} = \frac{p}{2y} = m, \; \frac{dy}{dx} = \frac{2x}{p'} = n,$$

and the preceding four equations become

$$y_0^2 - p(x_0 - a) = 0, \quad x_1^2 - p'(y_1 - b) = 0,$$

$$1 + \frac{p}{2y_0} \cdot \frac{y_1 - y_0}{x_1 - x_0} = 0, \; 1 + \frac{2x_1}{p'} \cdot \frac{y_1 - y_0}{x_1 - x_0} = 0;$$

from which the values of x_0, y_0, x_1, y_1 may be determined.

b. But this particular example may otherwise be solved as follows.

Since the line is straight, its equation is

$$y - y_0 = \frac{y_1 - y_0}{x_1 - x_0}(x - x_0);$$

and the expression for its length between the given curves is $\sqrt{(x_1 - x_0)^2 + (y_1 - y_0)^2}$, which is to be a minimum.

Differentiating, we have

$$(x_1 - x_0)(dx_1 - dx_0) + (y_1 - y_0)(dy_1 - dy_0) = 0.$$

Now the equations to the parabolas,

$$x^2 = p'(y - b), \quad y^2 = p(x - a),$$

give, for the points required,

$$dy_1 = \frac{2x_1}{p'} dx_1, \; dy_0 = \frac{p}{2y_0} dx_0 ;$$

hence we have

$$\left\{ (x_1 - x_0) + (y_1 - y_0) \frac{2x_1}{p'} \right\} dx_1 - \left\{ (x_1 - x_0) + (y_1 - y_0) \frac{p}{2y_0} \right\} dx_0 = 0;$$

but dx_1, dx_0 are entirely independent of each other, therefore this equation can only be satisfied by equating each coefficient to 0. Hence we have

$$(x_1 - x_0) + (y^1 - y_0) \frac{2x_1}{p'} = 0, \; (x_1 - x_0) + (y_1 - y_0) \frac{p}{2y_0} = 0;$$

or,

$$1 + \frac{y_1 - y_0}{x_1 - x_0} \cdot \frac{2x_1}{p'} = 0, \; 1 + \frac{y_1 - y_0}{x_1 - x_0} \cdot \frac{p}{2y_0} = 0:$$

which are the equations found before.

EXAMPLE IV.

To determine the brachystochrone, or curve of quickest descent from one given point to another.

The curve being referred to three rectangular axes, of which those of y and z are horizontal, and that of x vertical, or in the direction of gravity: let x, y, z be the co-ordinates of any point, s the arc described, and t the corresponding time; then the velocity acquired at this point will be the velocity due to the height x; representing by g, the force of gravity, we have

$$dt = \frac{ds}{\sqrt{2gx}} = \sqrt{\frac{1+y'^2+z'^2}{2gx}}\, dx,$$

the integral of which must be a minimum.

$$\therefore U = \int_{x_0}^{x_1} \sqrt{\frac{1+y'^2+z'^2}{2gx}}\, dx,$$

and

$$V = \sqrt{\frac{1+y'^2+z'^2}{2gx}}.$$

Hence the equations $H = 0$, $K = 0$, art. (22), p. 55 become

$$P' = 0, \qquad p' = 0;$$

integrating,

$$P = C, \qquad p = C',$$

Now

$$P = \frac{dV}{dy'} = \frac{y'}{\sqrt{2gx(1+y'^2+z'^2)}},$$

$$p = \frac{dV}{dz'} = \frac{z'}{\sqrt{2gx(1+y'^2+z'^2)}};$$

hence we have, for the differential equations of the first order of the curve sought,

$$\frac{y'}{\sqrt{2gx(1+y'^2+z'^2)}} = C, \quad \frac{z'}{\sqrt{2gx(1+y'^2+z'^2)}} = C':$$

these equations give

$$Cz' - C'y' = 0;$$

multiplying by dx, putting dz for $z'dx$, dy for $y'dx$ and integrating, we have

$$Cz - C'y = C'';$$

C'' being a third arbitrary constant.

This is the equation to a plane, which is perpendicular to that of yz, since the vertical ordinate x is not contained in it. Hence we learn that the curve required is a plane curve, and comprised in a vertical plane. As the plane of xy is vertical and its position arbitrary, we shall suppose this plane to pass through the given points, and the origin of the co-ordinates to coincide with the higher of these points. We shall then have $x_0 = 0$, $y_0 = 0$, $z = 0$, $z' = 0$; and the differential equation to the curve is

$$\frac{y'}{\sqrt{2\,g\,x\,(1 + y'^2)}} = C;$$

replacing y' by its value $\frac{d\,y}{d\,x}$, and putting γ for $\frac{1}{2\,C^2 g}$, it becomes

$$d\,y = \frac{x\,d\,x}{\sqrt{\gamma\,x - x^2}};$$

and, integrating, we have

$$y = -\sqrt{\gamma\,x - x^2} + \frac{\gamma}{2} \cos^{-1} \frac{\gamma - 2\,x}{\gamma};$$

the arbitrary constant vanishing, since we have at once, $y = 0$, $x = 0$, at the commencement of the motion. This is the equation to a cycloid whose base is horizontal, and which passes through the higher of the given points, its vertex being downwards, and whose height or diameter of whose generating circle is the arbitrary constant γ.

To determine this constant, we may observe that the cycloid must pass through the lower of the given points; therefore the co-ordinates x_1, y_1 must satisfy the preceding equation: hence, the equation $X_1 = 0$, (21.), becomes

$$y_1 = -\sqrt{\gamma x_1 - x_1^2} + \frac{\gamma}{2} \cos^{-1} \frac{\gamma - 2x_1}{\gamma}:$$

from which γ may be found with the assistance of trigonometrical tables.

If we make the axis of x horizontal, and that of y vertical, we shall have

$$U = \int_0^{x_1} \sqrt{\frac{1 + y'^2}{2gy}}\, dx;$$

$$\therefore V = \sqrt{\frac{1 + y'^2}{2gy}},$$

and as V contains only y and y', we have, (23.),

$$V = Py' + C,$$

or,

$$\sqrt{\frac{1 + y'^2}{2gy}} = \frac{y'^2}{\sqrt{2gy(1 + y'^2)}} + C,$$

from which we obtain

$$\sqrt{2gy(1 + y'^2)} = \frac{1}{C};$$

$$\therefore \frac{1}{y'} \text{ or } \frac{dx}{dy} = \sqrt{\frac{y}{\frac{1}{2C^2g}y - y^2}},$$

whence,

$$dx = \frac{y\,dy}{\sqrt{\gamma y - y^2}};$$

which agrees with the preceding.

In order that the integral of $\dfrac{\sqrt{1+y'^2+z'^2}}{\sqrt{2gx}}\,dx$

may be a minimum, it is necessary that

$$\int_{x_0}^{x_1}\left(\frac{d^2V}{dy'^2}\delta y'^2 + 2\,\frac{d^2V}{dy'dz'}\delta y'\,\delta z' + \frac{d^2V}{dz'^2}\delta z'^2\right)dx$$

should be positive (19). p. 50

Now,

$$V = \frac{\sqrt{1+y'^2+z'^2}}{\sqrt{2gx}};$$

$$\therefore \frac{d^2V}{dy'^2} = \frac{1+z'^2}{\sqrt{2gx}\,(1+y'^2+z'^2)^{\frac{3}{2}}},$$

$$\frac{d^2V}{dy'dz'} = \frac{y'z'}{\sqrt{2gx}\,(1+y'^2+z'^2)^{\frac{3}{2}}},$$

$$\frac{d^2V}{dz'^2} = \frac{1+y'^2}{\sqrt{2gx}\,(1+y'^2+z'^2)^{\frac{3}{2}}}.$$

Hence,

$$\int_{x_0}^{x_1}\left(\frac{d^2 V}{dy'^2}\delta y'^2 + 2\,\frac{d^2 V}{dy'\,dz'^2}\,\delta y'\,\delta z' + \frac{d^2 V}{dz'^2}\,\delta z'^2\right)dx$$

$$=\int_{x_0}^{x_1}\frac{(1+z'^2)\,\delta y'^2+2y'z'\,\delta y'\,\delta z' + (1+y'^2)\,\delta z'^2}{\sqrt{2gx}\,(1+y'^2+z'^2)^{\frac{3}{2}}}\,dx$$

$$=\int_{x_0}^{x_1}\frac{\delta y'^2+\delta z'^2 + (z'\,\delta y' + y'\,\delta z')^2}{\sqrt{2gx}\,(1+y'^2+z'^2)^{\frac{3}{2}}}\,dx,$$

a quantity essentially positive, whatever be the signs of $\delta y'$ and $\delta z'$.

EXAMPLE V.

Of all cycloids whose bases are horizontal that can be drawn between a given point and a given curve, to determinate that of quickest descent.

Let the axis of x be horizontal, that of y vertical; and since the first limit is fixed, δx_0 and δy_0 vanish, and the equation of limits

$$\Gamma = 0,$$

becomes simply,

$$V_1\,\delta x_1 + P_1\,\omega_1 = 0,$$

or

$$P_1\,\delta y_1 + (V_1 - P_1\,y_1)\,\delta x_1 = 0;$$

let $B = 0$ be the equation to the given curve; then from the variation of $B=0$, we obtain $\delta y = n\, \delta x$. Substituting for δy_1, P_1, V_1, their respective values,

$$n_1\, \delta x_1, \frac{y'_1}{\sqrt{2 g y_1 (1 + y'^2_1)}}, \sqrt{\frac{1 + y'^2_1}{2 g y_1}},$$

in the preceding equation, reducing and omitting the common factor δx_1 we obtain

$$\frac{n_1 y'_1 + 1}{\sqrt{2 g y (1 + y'^2_1)}} = 0;$$

$$\therefore n_1 y'_1 + 1 = 0,$$

which shows that the cycloid of quickest descent, must cut the given curve at right angles.

EXAMPLE VI.

Of all the cycloids that can be drawn between two given curves, to find the position of that of quickest descent.

Let $A = 0$, $B = 0$ be the equations to the given curves which are the loci of the limits of the required cycloid; from which we may deduce,

$$\delta y = m\, \delta x, \qquad \delta y = n\, \delta x:$$

the equation $\Gamma=0$, becomes

$$P_1\delta y_1+(V_1-P_1y'_1)\delta x_1-P_0\delta y_0-(V_0-P_0y'_0)\delta x_0=0,$$

or

$$(P_1m_1+V_1-P_1y'_1)\delta x_1-(P_0m_0+V_0-P_0y'_0)\delta x_0=0;$$

this equation, since δx_1, δx_0 are independent of each other, separates into the two equations,

$$P_1m_1+V_1-P_1y'_1=0, \qquad P_0m_0+V_0-P_0y'_0=0:$$

proceeding as in the last Example, we find

$$m_1y'_1+1=0, \qquad m_0y'_0+1=0;$$

therefore the cycloid which satisfies the condition required in the question, must cut both the given curves at right angles.

If there exist several cycloids, having horizontal bases, whose common summit or origin is at the same point, and which cut the given curves at right angles; we must conclude that the question admits of as many *minima* times of descent. These again separate into maxima and minima or cycloids of quickest and slowest descent.

(26.) In problems such as the preceding, we determine the arbitrary constants by making the curve pass through the given points; or by causing its direction to be regulated in a given manner at the limits of the integral: these constants may thus be considered functions of the co-ordinates of those points; hence the equation to the required curve will, generally, be of the form,

$$y = F(x, x_0, y_0, x_1, y_1).$$

If we substitute in V the value of y and the corresponding values of y', y'', &c., the quantity V will, likewise, become a function of x, x_0, y_0, x_1, y_1; and if we take afterwards the integral of $V\,dx$, from $x = x_0$ to $x = x_1$, we shall have resulting a certain function (T) of x_0, y_0, x_1, y_1. Now the value of this definite integral will depend upon the positions of the limits; and we shall obtain its maxima and minima values by taking its complete differential with respect to the four quantities x_0, y_0, x_1, y_1, and equating the expression to zero.

The maxima and minima times of descent mentioned in the preceding paragraph correspond to the maxima and minima values of the function T. We may distinguish the one from the other by the sign of the second differential d^2T. The cycloids which correspond to minima or to positive values of d^2T will be those of quickest descent; whilst those which correspond to maxima or to negative values of

d^2T will be *relatively* cycloids of the slowest descent.

EXAMPLE VII.

To find the curve of quickest descent situated between two given curves; the motion being supposed to commence at a point whose distance from the axis is a function of the co-ordinates of the first limit.

Let h be the given function of x_0, y_0 which represents the distance of the point where the motion commences from the horizontal axis, which we shall suppose to be that of x: then the velocity at the point whose co-ordinates are x, y will be the velocity due to the height $y - h$: hence we have

$$U = \int_{x_0}^{x_1} \sqrt{\frac{1 + y'^2}{2g(y - h)}} dx, V = \sqrt{\frac{1 + y'^2}{2g(y - h)}};$$

proceeding as in the second solution of Example IV., we find

$$dx = \frac{(y - h)\, dy}{\sqrt{\gamma (y - h) - (y - h)^2}},$$

the differential equation to the cycloid; whence by integrating,

$$x + C = -\sqrt{\gamma(y-h)-(y-h)^2} + \frac{\gamma}{2}\cos^{-1}\frac{\gamma-2(y-h)}{\gamma},$$

C being an arbitrary constant.

Now since x_0, y_0 are implicitly contained in V, the expression Γ will contain, in addition to the quantity

$$P_1\,\delta y_1 + (V_1 - P_1 y'_1)\,\delta x - P_0\,\delta y_0 - (V_0 - P_0 y'_0)\,\delta x_0,$$

the terms,

$$\delta x_0 \int_{x_0}^{x_1} \frac{dV}{dx_0}\,dx,\quad \delta y_0 \int_{x_0}^{x_1} \frac{dV}{dy_0}\,dx,\ (17.),$$ p. 48.

which arise from the variations of x_0, y_0 contained in h.

And since y and h are symmetrical in V,

$$\frac{dV}{dh} = -\frac{dV}{dy} = -N;$$

$$\therefore \frac{dV}{dx_0} = \frac{dV}{dh}\cdot\frac{dh}{dx_0} = -N\frac{dh}{dx_0},$$

$$\frac{dV}{dy_0} = -N\frac{dh}{dy_0}.$$

Hence we have

$$\delta x_0 \int_{x_0}^{x_1} \frac{d\,V}{d\,x_0}\,d\,x = -\delta x_0 \int_{x_0}^{x_1} N \frac{d\,h}{d x_0}\,d\,x$$

$$= -\frac{d\ h}{d x_0}\,\delta\,x_0 \int_{x_0}^{x_1} N\,d\ x,$$

since x is not contained in h:

$$\delta y_0 \int_{x_0}^{x_1} \frac{d\,V}{d\,y_0}\,d\,x = -\frac{d\,h}{d y_0}\ \delta\,y_0 \int_{x_0}^{x_1} N\,d\,x\,;$$

but from the equation H $= 0$, or $N - \dfrac{d\,(P)}{d\,x} = 0$, we have

$$\int_{x_0}^{x_1} N\,d\,x = P_1 - P_0\,;$$

$$\therefore\quad x_0 \int_{x_0}^{x_1} \frac{d\ V}{d\,x_0}\,d\,x + \delta\,y_0 \int_{x_0}^{x_1} \frac{d\ V}{d\,y_0}\,d\,x = \left(\frac{d\ h}{d x_0}\,\delta\,x_0 + \frac{d\,h}{d\,y_0}\,\delta\,y_0\right) (P_0 - P_1$$

Hence the equation $\Gamma = 0$ becomes

$$P_1\,\delta y_1 + (V_1 - P_1\,y'_1)\,\delta x_1 - \left(V_0 - P_0\,y'_0 - \frac{d\ h}{d\,x_0}(P_0 - P_1)\right)\delta\,x_0$$

$$-\left(P_0 - \frac{d\ h}{d\,y_0}(P_0 - P_1)\right)\delta\,y_0 = 0\,:$$

which, since $\delta\,x_0$ and $\delta\,y_0$ are totally independent of

δx_1, may be decomposed into the two equations of limits,

$$\left.\begin{array}{c} P_1 \delta y_1 + (V_1 - P_1 y'_1)\,\delta x_1 = 0, \\ V_0 - P_0 y'_0 - \frac{dh}{dx_0}(P_0 - P_1)\Big)\,\delta x_0 + \Big(P_0 - \frac{dh}{dy_0}(P_0 - P_1)\Big)\,\delta y_0 = 0 \end{array}\right\} (1).$$

If $h = y_0$ or $= y_0 - e$, e being a constant quantity; that is, if $y - h$, when the motion commences, $= 0$ or $= e$; or, if the origin of the coordinates be situated on the first curve, or the motion commence at the first curve, with an initial velocity, either given or equal to 0; we shall have

$$\frac{dh}{dy_0} = 1, \ \frac{dh}{dx_0} = 0;$$

and the equations (1) become

$$P_1 \,\delta y_1 + (V_1 - P_1 y'_1)\,\delta x_1 = 0,$$

$$(V_0 - P_0 y'_0)\,\delta x_0 + P_1 \,\delta y_0 = 0.$$

Substituting for δy_1, δy_0 their values $n_1 \delta x_1$, $m_0 \delta x_0$, drawn from the equations to the given curves, and remembering that $V_1 - P_1 y'_1 = C = V_0 - P_0 y'_0$, we deduce the equation

$$m_0 = n_1.$$

Whence it appears that the tangents to the given curves at the points in which the cycloid meets them, are parallel to one another; a conclusion which was first arrived at by Borda and afterwards confirmed by Legendre and Lagrange.

It was in the solution of this problem that Lagrange, by omitting to take the variation of the ordinate of the point where the motion commences, drew conclusions, which, when applied to the known solutions of particular cases of the Brachystochrone, were found to be not general. This gave rise to doubts respecting the exactness of the equation $\Gamma = 0$, which relates to the limits of the integral. Borda was the first who detected Lagrange's omission; and, supplying it, he cleared up the difficulty in a memoir which he published in the Mém. de l'Acad. des Sciences for the year 1767, soon after the discovery of the Calculus of Variations.

EXAMPLE VIII.

Required the relation between x and y, so that $\int (x^2 + y^2)^n \, d s$ may be a maximum.

Here,

$$U = \int (x^2 + y^2)^n \sqrt{1 + y'^2} \, d x,$$

$$V = (x^2 + y^2)^n \sqrt{1 + y'^2};$$

$$\therefore N = \frac{d V}{d y} = 2 n y \, (x^2 + y^2)^{n-1} \sqrt{1 + y'^2},$$

$$P = \frac{d V}{d y'} = \frac{(x^2 + y^2)^n y'}{\sqrt{1 + y'^2}};$$

and since

$$P' \text{ or } \frac{d(P)}{d x} = \frac{d P}{d x} + \frac{d P}{d y} \cdot \frac{d y}{d x} + \frac{d P}{d y'} \cdot \frac{d y'}{d x}$$

$$= \frac{(x^2 + y^2)^{n-1}}{\sqrt{1 + y'^2}} \left(2 n y' x + 2 n y y'^2 + \frac{x^2 + y^2}{1 + y'^2} y'' \right),$$

the equation $H = 0$ or $N - P' = 0$, becomes

$$2\,n\,y\,(x^2+y^2)^{n-1}\sqrt{1+y'^2}$$

$$-\frac{(x^2+y^2)^{n-1}}{\sqrt{1+y'^2}}\left(2\,n\,x\,y'+2\,n\,y\,y'^2+\frac{x^2+y^2}{1+y'^2}y''\right)=0,$$

omitting the common factor $(x^2+y^2)^{n-1}$, and multiplying by $\sqrt{1+y'^2}$, we have

$$2\,n\,y\,(1+y'^2)-2\,n\,x\,y'-2\,n\,y\,y'^2-\frac{x^2+y^2}{1+y'^2}y''=0;$$

whence we obtain

$$\frac{2\,n\,(y-x\,y')}{x^2+y^2}=\frac{y''}{1+y'^2},$$

or

$$\frac{2n\,(y\,d\,x-x\,d\,y)}{x^2+y^2}=\frac{d\,y'}{1+y'^2}.$$

Assume $\theta=\tan^{-1}\frac{x}{y}$, then

$$d\,\theta=\frac{d\,.\,\frac{x}{y}}{1+\frac{x^2}{y^2}}=\frac{y\,d\,x-x\,d\,y}{x^2+y^2}:$$

$$\therefore 2\,n\,d\,\theta=\frac{d\,y'}{1+y'^2},$$

and, integrating,

$$2 n \theta = \tan^{-1} y' + c;$$

c being an arbitrary constant.

Since

$$\frac{x}{y} = \tan \theta,$$

if we substitute for θ its value, we shall have

$$\frac{x}{y} = \tan \left\{ \frac{1}{2n} \left(\tan^{-1} \frac{dy}{dx} + c \right) \right\}.$$

EXAMPLE IX.

Required the curve in which $\int x y \, d s$ is a maximum.

Here,

$$U = \int x y \sqrt{1 + y'^2} \, d x,$$

$$V = x y \sqrt{1 + y'^2};$$

and since V contains x, y and y', we must make use of the form,

$$V = \int M \, d x + P y' + C.$$

Now,

$$M = \frac{d V}{d x} = y \sqrt{1 + y'^2}, \; P = \frac{d V}{d y'} = \frac{x y y'}{\sqrt{1 + y'^2}};$$

$$\therefore x y \sqrt{1 + y'^2} = \int y \sqrt{1 + y'^2} \, d x + \frac{x y y'^2}{\sqrt{1 + y'^2}} + C,$$

or

$$\frac{x y}{\sqrt{1 + y'^2}} = C + \int y \sqrt{1 + y'^2} \, d x,$$

which leads to a complicated differential equation of the second order.

EXAMPLE X.

Required the relation of x and y, or the curve in which $\int \frac{d^2 y}{d s}$ is a maximum.

In this example,

$$U = \int \frac{d^2 y}{\sqrt{1+y'^2}\cdot dx} \cancel{dx} = \int \frac{d^2 y}{d x^2} \cdot \frac{1}{\sqrt{1+y'^2}} d x$$

$$= \int \frac{y''}{\sqrt{1+y'^2}} d x ;$$

$$\therefore V = \frac{y''}{\sqrt{1+y'^2}},$$ which contains y' and y'';

and since

$$Q = \frac{d V}{d y''} = \frac{1}{\sqrt{1+y'^2}},$$

the equation

$$V = Q y'' + C y' + C',$$

becomes

$$\frac{y''}{\sqrt{1+y'^2}} = \frac{y''}{\sqrt{1+y'^2}} + C y' + C' ;$$

$$\therefore y' = \text{constant},$$

the differential equation to a straight line.

EXAMPLE XI.

Required the curve in which $\int \frac{d^2 y}{y\,d s}$ is a maximum.

Here,

$$U = \int \frac{d^2 y}{y\sqrt{1+y'^2}\,dx}\cancel{dx} = \int \frac{d^2 y}{d x^2} \cdot \frac{d x}{y\sqrt{1+y'^2}};$$

$$\therefore V = \frac{y''}{y\sqrt{1+y'^2}},$$

into which y, y' and y'' enter.

Hence we must use the equation,

$$V = P y' - Q' y' + Q y'' + C.$$

Now,

$$P = -\frac{y' y''}{y(1+y'^2)^{\frac{3}{2}}},\ Q = \frac{1}{y\sqrt{1+y'^2}},$$

$$Q' = \frac{d(Q)}{d x} = \frac{d Q}{d y} \cdot \frac{d y}{d x} + \frac{d Q}{d y'} \cdot \frac{d y'}{d x}$$

$$= -\frac{y'}{y^2\sqrt{1+y'^2}} - \frac{y' y''}{y(1+y'^2)^{\frac{3}{2}}},$$

$$\therefore \cancel{\frac{y''}{y\sqrt{1+y'^2}}} = -\cancel{\frac{y'^2 y''}{y(1+y'^2)^{\frac{3}{2}}}}$$

$$+\frac{y'^2}{y^2\sqrt{1+y'^2}} + \cancel{\frac{y'^2 y''}{y(1+y'^2)^{\frac{3}{2}}}} + \cancel{\frac{y''}{y\sqrt{1+y'^2}}} + C.$$

Hence we have

$$\frac{y'^2}{y^2\sqrt{1+y'^2}} = C,$$

from which we deduce,

$$y'^2 = \frac{Cy^2}{2}\left(Cy^2 \pm \sqrt{4+C^2y^4}\right)$$

$$\therefore y' \text{ or } \frac{dy}{dx} = y\sqrt{\left\{\frac{C}{2}(Cy^2 \pm \sqrt{4+C^2y^4})\right\}},$$

and

$$dx = \frac{\sqrt{\frac{2}{C}}\,.\,dy}{y\sqrt{(Cy^2 \pm \sqrt{4+C^2y^4})}}.$$

EXAMPLE XII.

Required the curve which by the revolution about its axis generates the solid of the least resistance.

By the principles of Mechanics, the resistance of a molecule of the fluid against the surface, varying as the square of the sine of inclination, may be represented by $\alpha \left(\frac{dy}{ds}\right)^2$; and since the whole number of molecules acting upon the body, is proportional to the area of a circle whose radius is y, that is to y^2; the differential of which, multiplied by $\alpha \left(\frac{dy}{ds}\right)^2$, will be proportional to the differential of the resistance.

Hence we have

$$U = \int \rho \frac{y y'^3}{1 + y'^2} dx:$$

ρ being a constant quantity depending upon π, and the velocity of the body, and the nature of the resisting medium.

Now,

$$V = \frac{\rho\, y\, y'^3}{1+y'^2},$$

$$\therefore\ P = \frac{3\,\rho\, y\, y'^2 + \rho\, y\, y'^4}{(1+y'^2)^2},$$

and the equation $V = P\,y' + C$, becomes p. 59, 1st.

$$\frac{\rho\, y\, y'^3}{1+y'^2} = \frac{3\,\rho\, y\, y'^3 + \rho\, y\, y'^5}{(1+y'^2)^2} + C;$$

$$\therefore\ \rho\, y\, y'^3 + \rho\, y\, y'^5 = 3\,\rho\, y\, y'^3 + \rho\, y\, y'^5 + C\,(1+y'^2)^2,$$

or

$$2\,\rho\, y\, y'^3 + C\,(1+y'^2)^2 = 0;$$

whence, by putting $-\,a$ for $\frac{C}{2\,\rho}$, we have

$$y = \frac{a\,(1+y'^2)^2}{y'^3} = \frac{a}{y'^3} + \frac{2\,a}{y'} + a\,y' \ldots (1).$$

But

$$y' = \frac{d\,y}{d\,x}\ \therefore\ d\,x = \frac{d\,y}{y'};$$

substituting for $d\,y$ its value deduced from equation (1) by differentiation, we have

$$dx = -3a\frac{dy'}{y'^5} - \frac{2a\,dy'}{y'^3} + \frac{a\,dy'}{y'};$$

$$\therefore\ x = \frac{3a}{4y'^4} + \frac{a}{y'^2} + a\log y' + C \ldots (2).$$

Now if y' were eliminated between the equations (1) and (2), the relation between x and y, or the equation to the generating curve would be obtained.

It appears, from an equation in the preceding page, that the curve does not meet the axis of x; for

$$y = \frac{a(1+y'^2)^2}{y'^3} = \frac{a\,ds^4}{dy^3\,dx},$$

and the numerator being always greater than the denominator, shews that y is always greater than a.

EXAMPLE XIII.

To find the curve or the relation of x and y when $\int \frac{(d^2 y)^2}{d x^3}$ is a maximum.

Here,

$$U = \int \frac{(d^2 y)^2}{d x^3} = \int \left(\frac{d^2 y}{d x^2}\right)^2 d x = \int y''^2 \, d x;$$

$$\therefore \; V = y''^2, \; Q = \frac{d V}{d y''} = 2 y''.$$

Hence the equation H = 0, becomes

$$\frac{d^2 (Q)}{d x^2} = 0, \quad \text{or } 2 \frac{d^2 y''}{d x^2} = 0;$$

integrating successively,

$$2 \frac{d y''}{d x} = C;$$

$$2 y'' \text{ or } 2 \frac{d y'}{d x} = C x + C';$$

$$2 y' \text{ or } 2 \frac{d y}{d x} = \frac{C x^2}{2} + C' x + C'';$$

3 / $$2 y = \frac{C x^6}{2 . 3} + \frac{C' x^2}{1 . 2} + \frac{C'' x}{1} + C''':$$

$$\therefore\ y = \tfrac{1}{2}\left\{\frac{C}{1.2.3}x^3 + \frac{C'}{1.2}x^2 + \frac{C''}{1}x + C'''\right\}.$$

In this example V is a differential function of the second order, and the complete integral of the equation H $= 0$, contains four arbitrary constants. To determine these constants we must have additional data. If we suppose that $y = 0$ when $x = 0$, and $y = b$ when $x = a$: also that the curve, at these points, is inclined to the axis of x at angles whose tangents are t, t', we shall have

$$C''' = 0,$$

$$b = \tfrac{1}{2}\left\{\frac{C}{1.2.3}a^3 + \frac{C'}{1.2}a^2 + \frac{C''}{1}a\right\};$$

$$\frac{dy}{dx},\ \text{when } x = 0,\ = t\ \therefore\ t = \tfrac{1}{2}C'';$$

$$\frac{dy}{dx},\ \text{when } x = a,\ = t'\ \therefore\ t' = \tfrac{1}{2}\left(\frac{C}{2}a^2 + C'a + C''\right).$$

$$\therefore\ C''' = 0,\ C'' = 2t,\ C' = \frac{12b - 4at' - 8at}{a^2}$$

$$C = \frac{12a(t' - t) - 24b}{a^3}.$$

EXAMPLE XIV.

Required the relation between x and y so that the function $\int (y'^2+2\,m\,y\,y'+n\,y^2)\,d\,x$ may be a minimum.

Here

$$V = y'^2+2\,m\,y\,y'+n\,y^2,$$

and since,

$$N=2\,(m\,y'+n\,y),\ P=\frac{d\,V}{d\,y'}=2\,(y'+m\,y),$$

the equation H $=0$ or $N-\dfrac{d\,(P)}{d\,x}=0$, becomes

$$2\,(m\,y'+n\,y)-2\,(y''+m\,y')=0\,;$$

$$\therefore\ y''-n\,y=0,$$

or

$$\frac{d^2\,y}{d\,x^2}-n\,y=0\,;$$

$$\therefore\ y=g\,e^{x\sqrt{n}}+h\,e^{-x\sqrt{n}},$$

e being the base of the Naperian logarithms and g and h two arbitrary constants.

If a be negative and equal to $-k^2$, then we shall have

$$y = g \sin (k x + h),$$

g and h being two other arbitrary constant quantities *.

EXAMPLE XV.

Required the curve which within its arc, the curve of its evolute, and the radius of curvature at its extremity, shall contain the least area.

Let R be the radius of curvature at any point, s the corresponding arc measured from the vertex; then since

$$R = \frac{(1+y'^2)^{\frac{3}{2}}}{-y''}, \quad s = \int \sqrt{1+y'^2}\, d x,$$

the differential of the area $= \dfrac{R\, d s}{2}$

$$= -\frac{(1+y'^2)^{\frac{3}{2}} \sqrt{1+y'^2}}{2\, y''} d x$$

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* Lacroix, Tom. II. p. 320. Airy's Mathematical Tracts, Int.

$$= -\frac{(1+y'^2)^2}{2y''}\,dx.$$

$$\therefore\ U = \int \frac{(1+y'^2)^2}{2y''}\,dx,$$

$$V = \frac{(1+y'^2)^2}{2y''}.$$

Hence the equation $V = Q\,y'' + C\,y' + C$ becomes p. 60.

$$\frac{(1+y'^2)^2}{2\,y''} = -\frac{(1+y'^2)^2}{2\,y''} + C\,y' + C',$$

or

$$\frac{2\,(1+y'^2)^2}{y''} = C\,y' + C'';$$

but $y'' = \dfrac{d\,y'}{d\,x}$;

$$\therefore\ 2\,(1+y'^2)^{\frac{3}{2}}\,\sqrt{1+y'^2}\,d\,x = C\,y'\,d\,y' + C'\,d\,y',$$

or

$$2\,d\,s = \frac{C\,y'\,d\,y'}{(1+y'^2)^{\frac{3}{2}}} + \frac{C''\,d\,y'}{(1+y'^2)^{\frac{3}{2}}};$$

whence by integration,

$$2\,(s - C'') = \frac{C\,y' - C'}{\sqrt{1+y'^2}},$$

or

$$2\,(s - C''')\,d\,s = C\,d\,y - C'\,d\,x,$$

$$\therefore (s - C'')^2 = C y - C' x + C'''.$$

If we place the origin at another point in the axis of x, we may assume $x = x + X$, and then making $- C' X + C''' = 0$ we get quit of the constant C'''.

Hence we have

$$s - C'' = \sqrt{C y - C' x},$$

the equation to the curve sought.

If $C'' = 0$, $C' = 0$, we have

$$s = \sqrt{C y},$$

and the curve is a cycloid, the diameter of whose generating circle is $\frac{C}{4}$, since the arc of the cycloid, measured from the vertex, $=$ twice the chord of the generating circle.

EXAMPLE XVI.

To find the curve in which $\int \frac{d^3 y}{d s\, d x}$ is a maximum.

Here we have

$$U = \int \frac{d^3 y}{d s\, d x} = \int \frac{d^3 y}{d x^3} \frac{d x}{\sqrt{1 + y'^2}} = \int \frac{y'''}{\sqrt{1 + y'^2}} d x,$$

and

$$V = \frac{y'''}{\sqrt{1 + y'^2}}, \text{ a function of } y' \text{ and } y'''.$$

$$\therefore R = \frac{d V}{d y'''} = \frac{1}{\sqrt{1 + y'^2}};$$

$$R' = \frac{d (R)}{d x} = \frac{d R}{d y'} \cdot \frac{d y'}{d x} = - \frac{y' y''}{(1 + y'^2)^{\frac{3}{2}}}.$$

Hence the equation

$$V = - R' y'' + R y''' + C y' + C, \quad (5)\ p.\ 61$$

becomes

$$\frac{y'''}{\sqrt{1+y'^2}} = \frac{y'\,y''^2}{(1+y'^2)^{\frac{3}{2}}} + \frac{y'''}{\sqrt{1+y'^2}} + Cy' + C'.$$

$$\therefore \ \frac{y'\,y''^2}{(1+y'^2)^{\frac{3}{2}}} + Cy' + C' = 0,$$

a differential equation of the second order and second degree.

SECTION VI.

RELATIVE MAXIMA AND MINIMA.

27. THE preceding problems involve only one property, viz., that of maximum or minimum; and this being satisfied, every other circumstance affecting the nature of the inquiry is indeterminate. Such problems have received the name of *absolute* maxima or minima.

But when to the maximum or minimum property, other conditions are added, that property becomes in consequence restricted, and modified so that the given conditions may also be satisfied. For example, when the question is to find the curve of quickest descent, the length being given; both the minimum and the given property must be incorporated into the nature of the required curve. These problems are denominated *relative* maxima or minima.

28. Let V, T, W, &c., be given functions of x, y, y', y'', &c. and supposing

$$v=\int_{x_0}^{x_1} V\,dx,\ t=\int_{x_0}^{x_1} T\,dx,\ w=\int_{x}^{x_1} W\,dx, \text{ \&c.}$$

in which v is a maximum or minimum, $t=l$, $w=m$, &c.; l, m, &c., being given quantities; we shall have at once,

$$\delta v=0,\ \delta t=0,\ \delta w=0, \text{ \&c.}\dots(1).$$

If we should determine the function y from the single condition $v=$ a maximum or minimum, the conditions $t=l$, $w=m$, &c., would not be satisfied, for the problem requires the equations (1) to exist simultaneously. But if we assume

$$\text{U}=v+a\,t+b\,w+\text{\&c.},$$

in which a, b, &c., are constant factors, it is evident that that value of y which makes the sum or new function U a maximum or minimum absolutely, or with respect to all possible values, will render its first term v a maximum or minimum relatively; that is with respect to the conditions $t=l$, $w=m$, &c., since the remaining terms, $a\,t$, $b\,w$, &c., are constant quantities. At the same time the arbitrary constants a, b, &c., enable us to satisfy the given conditions.

In fact when δv, δt, δw, &c., are functions of x, y, y', &c., and a, b, &c., are totally independent of them, the equation

$$\delta v + a\,\delta t + b\,\delta w + \text{\&c.} = 0,$$

really embraces the separate equations

$$\delta v = 0,\ \delta t = 0,\ \delta w = 0, \text{\&c.}$$

Hence the problem of relative maxima or minima is reduced to that of absolute maximum or minimum of the single integral

$$\mathrm{U} = \int_{x_0}^{x_1} (V + a\,T + b\,W + \text{\&c.})\,dx:$$

and the variation of U, equated to zero, being decomposed into the two equations $\Gamma = 0$, $\mathrm{H} = 0$, affords a complete solution to the problem.

29. The principle by which we reduce one problem to the other, may also be obtained by decomposing each integral into its infinitely small elements.

Let there be three integrals v, t, w, and suppose V_n, T_n, W_n to be the values of V, T, W when x is replaced by $x + n\,dx$; also let $i\,dx = x_1 - x_0$, i being infinite; then,

$$v=\int_{x_0}^{x_1} V\,dx=V_0\,dx+V_1\,dx+V_2\,dx\ldots.\ V_i\,dx,$$

$$t=\int_{x_0}^{x_1} T\,dx=T_0\,dx+T_1\,dx+T_2\,dx\ldots.\ T_i\,dx,$$

$$w=\int_{x_0}^{x_1} W\,dx=W_0\,dx+W_1\,dx+W_2\,dx\ldots.W_i\,dx.$$

Taking the variations of these equations arising from the arbitrary increments of y, and of x_0, x_1 at the limits, supposing $d\,x$ constant, we shall have

$$\begin{aligned}\delta v&=\delta V_0\,dx+\delta V_1\,dx+\delta V_2\,dx\ldots\ldots\delta V_i\,dx\\&\qquad+V_1\,\delta x_1-V_0\,\delta x_0,\\ \delta t&=\delta T_0\,dx+\delta T_1\,dx+\delta T_2\,dx\ldots\ldots\delta T_i\,dx\\&\qquad+T_1\,\delta x_1-T_0\,\delta x_0,\\ \delta w&=\delta W_0\,dx+\delta W_1\,dx+\delta W_2\,dx\ldots\ldots\delta W_i\,dx\\&\qquad+W_1\,\delta x_1-W_0\,\delta x_0.\end{aligned}$$

Now when v is an absolute maximum or minimum, $\delta v=0$; and the values of y between the given limits x_0 and x_1, having no connection from other conditions, are entirely independent of one another. Hence the coefficients of the variations of y in the equation $\delta v=0$, must separately be equal to 0. In this manner we deduce the differential equation which serves to determine y in terms of x, and a certain number of arbitrary constants.

But when v is to be a maximum or minimum, subject to the conditions $t=l$ and $w=m$, it is not necessary that δv should be equal to 0, for all values of y between the given limits, but only for such as make $\delta t=0$ and $\delta w=0$; so that we must have, simultaneously,

$$\delta v=0, \quad \delta t=0, \quad \delta w=0;$$

hence the variations of all the intermediate values of y will no longer be independent of one another in the equation $\delta v=0$: but these variations, equal in number to that of the equations $\delta v=0$, $\delta t=0$, $\delta w=0$, will have determinate values, which must be deduced from the latter two equations and substituted in the first; then equating to zero the coefficient of the variation not eliminated, we shall have the differential equation by which the function y is to be determined. The arbitrary constants introduced by integration, serve to give to t and w their required values, l and m. Instead of eliminating, we may, as was done in (22), multiply $\delta t=0$, $\delta w=0$ by the constants a, b, and then add the products to $\delta v=0$. Thus we shall have

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$$\delta v+a\,\delta t+b\,\delta w=0,$$

or

$$\delta\,(v+a\,t+b\,w)=0;$$

an equation in which the variations of the intermediate values of y, as in absolute maximum or minimum, are totally independent of one another. Consequently the problem proposed, in which the integral $\int_{x_0}^{x_1} \mathrm{V}\,d\,x$ is to be a maximum or minimum, and the integrals $\int_{x_0}^{x_1} T\,d\,x$, $\int_{x_0}^{x_1} W\,d\,x$, are to have given values, is reduced, as in the preceding article, to that of the absolute maximum or minimum of another integral, viz.

$$\int_{x_0}^{x_1} (\mathrm{V}+a\,T+b\,\mathrm{W})\,d\,x.$$

30. The geometers who first determined the curve corresponding to the maximum or minimum of an indeterminate integral in its whole extent, were satisfied with making the ordinate of a single point in the curve, or two of its elements, vary, and deducing therefrom, the corresponding variation of the integral in question, which they equated to zero. When the curve was further required to possess a given property, it was necessary to make two ordinates of the curve, or three consecutive elements, vary; from one of which an equation was obtained involving the maximum or minimum; and from the other an equation containing the given condition. These two equations involved two arbitrary

quantities; by their elimination, the curve was determined into which both properties were incorporated.

In general, if several integrals relative to the length, &c., of the curve of maximum or minimum have given values, and if the abscissa of any point be taken as the independent variable, it is necessary to make the co-ordinates of as many consecutive points of the curve vary simultaneously as are equal to the number of these integrals augmented by unity; then to equate to zero the total variation of each of the given integrals, and of that which is to be a maximum or minimum. By combining these equations we obtain a property which must be possessed by the curve required. It was by proceeding upon this principle that Euler first reduced the problem of isoperimeters, taken in its most general acceptation, to a question of absolute maximum or minimum.

Let us take, for example, the curve in which

$$v=\int_{x}^{x_1} \mathrm{V}\, d\, x = \text{a maximum or minimum,}$$

$$t=\int_{x_0}^{x_1} T\, d\, x = l,\ w=\int_{x_0}^{x_1} \mathrm{W}\, d\, x = m;$$

l and m being given.

And let us suppose the three co-ordinates y, y_1, y_2, which vary, to correspond to the abscissas

x, $x+dx$, $x+2\,dx$; then since V, T, W are given functions of x, y, y', &c., if we replace y, y', y'', &c., by their first, second, third, &c., differentials of the consecutive values of y, divided by dx, dx^2, dx^3, &c., and suppose the increments of y, y_1, y_2, to be denoted by θ, θ_1, θ_2; we may represent by

$$\begin{gathered} p\,\theta + p_1\,\theta_1 + p_2\,\theta_2, \\ q\,\theta + q_1\,\theta_1 + q_2\,\theta_2, \\ r\,\theta + r_1\,\theta_1 + r_2\,\theta_2, \end{gathered}$$

the corresponding variations of the three integrals v, t, w; the nine coefficients p, p_1, p_2, q_1, q_2, r, &c., being certain unknown functions of the values of y which have varied, and of the corresponding values of x.

Now since $y_1 = y + dy$, if we change x into $x+dx$ we shall have $y_2 = y_1 + dy_1$ whence it is evident that p_1, q_1, r_1 may be drawn from p, q, r, and p_2, q_2, r_2, from p_1, q_1, r_1, by the like substitution of $x+dx$ for x; hence we have

$$\begin{aligned} p_1 &= p + dp,\ p_2 = p_1 + dp_1, \\ &\qquad = p + 2\,dp + d^2p, \\ q_2 &= q + dq,\ q_2 = q_1 + dq_1, \\ &\qquad = q + 2\,dq + d^2q, \\ r_1 &= r + dr,\ r_2 = r_1 + dr_1, \\ &\qquad = r + 2\,dr + d^2r. \end{aligned}$$

By substituting these values and equating to zero the variations of the three integrals, according to the conditions of the problem, we shall have

$$(\theta+\theta_1+\theta_2)\ p + (\theta+2\,\theta_2)\ dp + \theta_2\, d^2 p=0,$$
$$(\theta+\theta_1+\theta_2)\ q + (\theta+2\,\theta_2)\ d\,q + \theta_2\, d^2 q=0,$$
$$(\theta+\theta_1+\theta_2)\ r + (\theta+2\,\theta_2)\ d\,r + \theta_2\, d^2 r=0.$$

Eliminating $\theta+\theta_1+\theta_2$, we have

$$(\theta+2.\theta_2)\left(\frac{dp}{p}-\frac{d\,q}{q}\right)+\theta_2\left(\frac{d^2 p}{p}-\frac{d^2 q}{q}\right)=0,$$

$$(\theta+2\theta_2,\left(\frac{dp}{p}-\frac{d\,r}{r}\right)+\theta_2\left(\frac{d^2 p}{p}-\frac{d^2 r}{q}\right)=0;$$

eliminating in its turn $\theta+2\,\theta_2$, omitting the common factor θ_2, and we obtain

$$\frac{p\,d^2 q-q\,d^2 p}{p\,d\,q-q\,dp}-\frac{p\,d^2 r-r\,d^2 p}{p\,d\,r-r\,d\,p}=0.$$

Now if we integrate this equation, calling c the arbitrary constant, we shall have

$$\log\,(p\,d\,q-q\,d\,p)-\log c\,(p\,d\,r-r\,d\,p)=0\,;$$
$$\therefore\ p\,d\,q-q\,d\,p-c\,(p\,d\,r-r\,d\,p)=0,$$

or, which is the same thing,

$$\frac{d\,p}{p}-\frac{d\,q-c\,d\,r}{q-c\,r}=0\,;$$

integrating again, — a being the arbitrary constant,

$$p + a\,(q - c\,r) = 0\,;$$

or putting b for $-\,a\,c$, we have

$$p + a\,q + b\,r = 0\,;$$

which is the equation of absolute maximum or minimum of the integral

$$\int_{x}^{x} (V + a\,T + b\,W)\,d\,x.$$

EXAMPLE I.

Required the curve of quickest descent between two given points, its length being given.

Here we have

$$v = \int_{x_0}^{x_1} V\,d\,x = \int_{x_0}^{x_1} \sqrt{\frac{1+y'^2}{y}}\,d\,x, \text{ a minimum;}$$

$$t = \int_{x^0}^{x_1} T\,d\,x = \int_{x_0}^{x_1} \sqrt{1+y'^2}\,d\,x = l.$$

$$.\ U = \int_{x_0}^{x^1} \left(\sqrt{\frac{1+y'^2}{y}} + a\,\sqrt{1+y'^2}\right)\,d\,x,$$

and

$$V = \left(\frac{1}{\sqrt{y}} + a\right)\sqrt{1+y'^2};$$

hence the equation $V = P\,y' + C$, (23), becomes

$$\left(\frac{1}{\sqrt{y}} + a\right)\sqrt{1+y'^2} = \left(\frac{1}{\sqrt{y}} + a\right)\frac{y'^2}{\sqrt{1+y'^2}} + C,$$

or

$$\left(\frac{1}{\sqrt{y}} + a\right)(1+y'^2) = \left(\frac{1}{\sqrt{y}} + a\right)y'^2 + C\sqrt{1+y'^2};$$

$$\therefore\ \frac{1}{\sqrt{y}} + a = C\sqrt{1+y'^2},$$

and by reduction,

$$d\,x = \frac{C\sqrt{y}\,d\,y}{\sqrt{\{1+2\,a\sqrt{y} + (a^2-C^2)\,y\}}}.$$

the differential equation to the curve.

This equation contains two arbitrary constants, a and C; its integral will contain a third.

Let us suppose

$$y = F(x, a, b, C);$$

then to determine these constants we have the three conditions stated in the question:

$$y_0 = F(x_0, a, b, C), \qquad y_1 = F(x_1, a, b, C),$$

$$\frac{1}{C}\int_{x_0}^{x_1} \sqrt{\frac{1+2a\sqrt{y}+a^2 y}{y}}\, dx = l.$$

If, instead of the length, the area had been given, we should have

$$V = \sqrt{\frac{1+y'^2}{y}} + a\,y;$$

and the equation $V = P\,y' + C$, in this case, becomes

$$\sqrt{\frac{1+y'^2}{y}} + a\,y = \frac{y'^2}{\sqrt{y\,(1+y'^2)}} + C,$$

or,

$$\frac{1}{\sqrt{y\,(1+y'^2)}} = C - a\,y,$$

whence by reduction we obtain

$$dx = \frac{(C-a\,y)\sqrt{y}\,.\,dy}{\sqrt{\{1-y\,(C-a\,y)^2\}}}.$$

EXAMPLE II.

Given the length of a curve between two given points, to find its nature when the area is a maximum.

Here

$$v = \int_{x_0}^{x_1} V\,d\,x = \int_{x_0}^{x_1} y\,d\,x, \text{ a maximum };$$

$$t = \int_{x_0}^{x_1} T\,d\,x = \int_{x^0}^{x_1} \sqrt{1+y'^2}\,d\,x = l.$$

$$\therefore \mathrm{V} = y + a\sqrt{1+y'^2},$$

and the equation $\mathrm{V} = P\,y' + C$ becomes

$$y + a\sqrt{1+y'^2} = \frac{a\,y'^2}{\sqrt{1+y'^2}} + C,$$

therefore,

$$y\sqrt{1+y'^2} + a = C\sqrt{1+y'^2},$$

or,

$$(C-y)\sqrt{1+y'^2} = a;$$

whence we obtain

$$dx = \frac{(C-y)\,dy}{\sqrt{a^2-(C-y)^2}},$$

$$\therefore\ C'-x = \sqrt{a^2-(C-y)^2};$$

or,

$$(C'-x)^2+(C-y)^2 = a^2,$$

the equation to a circle, the co-ordinates of whose centre are C', C, and its radius a.

EXAMPLE III.

To find a curve returning into itself that shall contain the greatest area, its length being given.

Employing the polar co-ordinates r and θ, we have

$$v = \int_0^{2\pi} \frac{r^2\,d\theta}{2},\ \text{a maximum},$$

$$t = \int_0^{2\pi} \sqrt{r^2\,d\theta^2+dr^2} = \int_0^{2\pi} \sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}\,d\theta = l.$$

$$\therefore\ \mathrm{U} = \int_0^{2\pi}\left(\tfrac{1}{2}r^2+a\sqrt{r^2+r'^2}\right)d\theta,$$

$$\mathrm{V} = \tfrac{1}{2}r^2+a\sqrt{r^2+r'^2},$$

observing that the quantities x, y, y' are replaced by θ, r, r'; hence the equation $V = Pr' + c$ becomes

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$$\tfrac{1}{2} r^2 + a\sqrt{r^2+r'^2} = \frac{a\,r'^2}{\sqrt{r^2+r'^2}} + c,$$

or,

$$(c - \tfrac{1}{2} r^2)\sqrt{r^2+r'^2} = a\,r^2;$$

whence we deduce

$$d\theta = \frac{(r^2 - 2c)\,dr}{r\sqrt{4a^2r^2 - (r^2-2c)^2}};$$

which may be integrated as follows.

Equating to zero the quantity within the radical, and calling h^2 and k^2, the roots of the equation, $r^4 - 4(a^2+c)\,r^2 + 4c^2 = 0$, the expression for $d\theta$ may be put under the form,

$$d\theta = \frac{(r^2 \mp hk)\,dr}{r\sqrt{(h^2-r^2)(r^2-k^2)}};$$

in which we must use the sign $+$, since the curve returning into itself, and the origin of the polar coordinates being situated within the figure, the angle θ increases indefinitely; and its differential can never be equal to 0 for any value of r, such as

$r = \sqrt{hk}$, comprised between h and k. We have, therefore,

$$d\theta = \frac{(r^2 + hk)\,dr}{r\sqrt{(h^2 - r^2)(r^2 - k^2)}}.$$

Let us assume

$$h^2 - r^2 = u^2(r^2 - k^2),$$

then

$$r = \sqrt{\frac{h^2 + k^2u^2}{1 + u^2}},$$

and

$$dr = \frac{(k^2 - h^2)\,u\,du}{(1 + u^2)^{\frac{3}{2}}\sqrt{h^2 + k^2u^2}}.$$

Substituting for $h^2 - r^2$, r^2 and dr their respective values, the expression for $d\theta$ will become

$$d\theta = \frac{du}{1 + u^2} + \frac{hk\,du}{h^2 + k^2u^2}.$$

Integrating, we have

$$\theta = \tan^{-1} u + \tan^{-1} \frac{ku}{h},$$

in which we introduce no arbitrary constant, since we may suppose θ to be reckoned from the radius vector h, in which case $u = 0$.

We have then

$$\tan\theta = \frac{u + \dfrac{k\,u}{h}}{1 - \dfrac{k\,u^2}{h}} = \frac{(h+k)\,u}{h - k\,u^2},$$

or, as it becomes by substituting for u its value

$$\sqrt{\frac{h^2 - r^2}{r^2 - k^2}},$$

$$\tan\theta = \frac{(h+k)\sqrt{\dfrac{h^2 - r^2}{r^2 + k^2}}}{h - k\,.\,\dfrac{h^2 - r^2}{r^2 - k^2}} = \frac{\sqrt{(h^2 - r^2)\,(r^2 - k^2)}.}{r^2 - h\,k}$$

Squaring and replacing $\tan^2\theta$ by its value $\dfrac{1 - \cos^2\theta}{\cos^2\theta}$, we have

$$\frac{1 - \cos^2\theta}{\cos^2\theta} = \frac{(h^2 - r^2)\,(r^2 - k^2)}{(r^2 - h\,k)^2};$$

whence we deduce

$$(r^2 - h\,k)^2 = (h - k)^2\,r^2\cos^2\theta\,;$$

$$\therefore\ r^2 - h\,k = \pm\,(h - k)\,r\cos\theta.$$

Taking the sign +, that r may be $= h$, when $\theta=0$, we shall have

$$r^2-(h-k)\,r\cos\theta = h\,k,$$

or,

$$r^2-(h-k)\,r\cos\theta + \tfrac{1}{4}(h-k)^2=\tfrac{1}{4}\,(h+k)^2;$$

which is the polar equation to a circle whose radius $=\frac{1}{2}\,(h+k)$, and whose centre is situated at the distance $\frac{1}{2}\,(h-k)$ from the origin of the polar co-ordinates. The arbitrary constant or diameter $h+k$ will be determined from the length of the circumference l, and the other constant $h-k$ remains indeterminate.

If the curve, instead of returning into itself, should be terminated by two fixed points, and such that the area of the sector comprised between the two radii vectores and the given length, may be a maximum, the preceding equation would still be that to the curve sought; but in this case there will remain no constant indeterminate. Let f and g, (fig. 6.), be the two radii vectores at the two extremities of the curve, l its length, and α the given angle of the sector; we have

$$\tfrac{1}{2}\,(h+k) = \frac{l}{\alpha};$$

and if we represent by δ the unknown angle between the radii f and h, and consequently by $\alpha+\delta$, the angle between the radii g and h; so that $\theta=0$ when $r=h$, $\theta=\delta$ when $r=f$, and $\theta=\alpha+\delta$ when $r=g$; we shall have

$$f^2-(h-k)f\cos\delta+\tfrac{1}{4}(h-k)^2=\frac{l^2}{\alpha^2},$$

$$g^2-(h-k)g\cos(\alpha+\delta)+\tfrac{1}{4}(h-k)^2=\frac{l^2}{\alpha};$$

and by the elimination of δ, we obtain the value of the arbitrary constant $h-k$. When $\alpha=2\pi$, the curve returns into itself, or the circle is complete, the preceding equations are identical, and $h-k$ is indeterminate as before.

EXAMPLE IV.

Given the surface of a solid of revolution to find its nature when the solid content is a maximum.

Let x be measured along the axis of revolution; then,

$$v=\pi\int_{x_0}^{x_1} y^2\,dx, \text{ a maximum};$$

$$t=2\pi\int_{x_0}^{x_1} y\sqrt{1+y'^2}\,dx=l.$$

$$\therefore\ U=\pi\int_{x_0}^{x_1}\left(y^2+ay\sqrt{1+y'^2}\right)dx,$$

including the factor 2 in the constant a.
Hence

$$V=y^2+ay\sqrt{1+y'^2},$$

and the equation $V=Py'+C$, gives

$$\frac{ay}{\sqrt{1+y'^2}}=C-y^2,$$

from which we deduce

$$dx=\frac{(C-y^2)\,dy}{\sqrt{a^2y^2-(C-y^2)^2}}.$$

If $C=0$, we have simply, $dx=\dfrac{-y\,dy}{\sqrt{a^2-y'^2}}$;

whence the curve is a circle whose centre is in the axis of revolution.

EXAMPLE V.

Given the length of a curve to determine its nature when the solid generated by its rotation is a maximum.

Here,

$$v = \pi \int_{x_0}^{x_1} y^2\, d\,x, \text{ a maximum,}$$

$$t = \int_{x_0}^{x_1} \sqrt{1+y'^2}\, d\,x = l;$$

$$\therefore \mathrm{U} = \int_{x_0}^{x_1} \left(\pi\, y^2 + a\sqrt{1+y'^2}\right) d\,x,$$

$$\mathrm{V} = \pi\, y^2 + a\sqrt{1+y'^2};$$

hence we have

$$\pi\, y^2 + a\sqrt{1+y'^2} = \frac{a\,y'^2}{\sqrt{1+y'^2}} + C,$$

or

$$\pi y^2 \sqrt{} + a = a y'^2 + c\sqrt{1+y'^2}$$

$$(C - \pi\, y^2)\sqrt{1+y'^2} = a;$$

$$\pi y^2 \sqrt{} + a(1$$

C

whence

$$y'^2 = \frac{a^2 - (C - \pi\, y^2)^2}{(C - \pi\, y^2)^2},$$

$$\therefore d\,x = \frac{(C - \pi\, y^2)\, d\,y}{\sqrt{a^2 - (C - \pi\, y^2)^2}}.$$

EXAMPLE VI.

Required the curve that generates the least surface, the area being given.

Here

$$U=\int_{x_0}^{x_1}\left(2\pi y\sqrt{1+y'^2}+a y\right)dx,$$

$$\therefore\ V=2\pi y\sqrt{1+y'^2}+a y,$$

and we have

$$2\pi y\sqrt{1+y'^2}+a y=\frac{2\pi y y'^2}{\sqrt{1+y'^2}}+C,$$

from which we deduce

$$dx=\frac{(C-a y)\,dy}{\sqrt{\{4\pi^2 y^2-(C-a y)^2\}}}.$$

If the length of the curve be given instead of the area, we shall have

$$U=\int_{x_0}^{x_1}\left(2\pi y\sqrt{1+y'^2}+a\sqrt{1+y'^2}\right)dx;$$

hence,

$$V=(2\pi y+a)\sqrt{1+y'^2},$$

and the differential equation to the curve will be

$$d\,x = \frac{C\,d\,y}{\sqrt{\{(2\,\pi\,y + a)^2 - C^2\}}};$$

integrating, we have

$$x = \frac{C}{2\,\pi}\log\{2\,\pi\,y + a + \sqrt{(2\,\pi\,y + a)^2 - C^2}\} + C',$$

which is the equation to the catenary.

EXAMPLE VII.

To find the solid of least resistance among all solids of equal capacity.

Here

1. $$v = \int_{x_0}^{x_1} \frac{y\,y'^3}{1 + y'^2}\,d\,x, \text{ a minimum,}$$

$$t = \int_{x_0}^{x_1} \pi\,y^2\,d\,x = l;$$

$$\therefore\ U = \int_{x_0}^{x_1}\left(\rho\,\frac{y\,y'^3}{1 + y'^2} + a\,y^2\right)d\,x,$$

(including π in the constant a);

$$V=\frac{\rho y y'^3}{1+y'^2}+a y^2;$$

whence the equation $V=P y'+C$ becomes,

$$\frac{\rho y y'^3}{1+y'^2}+a y^2=\frac{3\rho y y'^3+\rho y y'^5}{(1+y'^2)^2}+C;$$

by reduction, we have

$$(a y^2-C^2)\ (1+y'^2)^2=2\rho y y'^3,$$

or

$$(a y^2-C^2)\ d s^4=2\rho y\ d y^3\ d x.$$

2. If, instead of equal capacity, equal surface be used, we shall have

$$V=\frac{\rho y y'^3}{1+y'^2}+a y\sqrt{1+y'^2},$$

and the equation $V=P y'+C$ will be

$$\frac{\rho y y'^3}{1+y'^2}+a y\sqrt{1+y'^2}=\frac{3c y y'^3+\rho y y'^5}{(1+y'^2)^2}+\frac{a y y'^2}{\sqrt{1+y'^2}}+C,$$

whence we deduce

$$C\,d s^4+2\rho y\,d y^3\,d x=a\,y\,d s^3\,d x.$$

3. If the solid of least resistance be required with

both the specified conditions, we shall have

$$U=\int_{x_0}^{x_1}\left(\frac{\rho\, y\, y'^3}{1+y'^2}+a\,y^2+b\,y\,\sqrt{1+y'^2}\right)d\,x,$$

and

$$V=\frac{\rho\, y\, y'^3}{1+y'^2}+a\,y^2+b\,y\,\sqrt{1+y'^2};$$

and we shall have

$$C\,(1+y'^2)^2+2\,\rho\, y\, y'^3=a\,y^2\,(1+y'^2)^2+b\,y\,(1+y'^2)^{\frac{3}{2}},$$

or

$$(C-a\,y^2)\,d\,s^4=b\,d\,s^3\,d\,x-2\,\rho\, y\,dy^3\,d\,x.$$

EXAMPLE VIII.

Required the curve which generates the solid of greatest volume, the length of the curve and its area being given.

In this example,

$$v = \int_{x_0}^{x_1} \pi y^2 \, dx, \text{ a maximum,}$$

$$t = \int_{x_0}^{x_1} \sqrt{1 + y'^2} \, dx = l,$$

$$w = \int_{x_0}^{x_1} y \, dx = m.$$

$$\therefore \ \mathrm{U} = \int_{x_0}^{x_1} \left(y^2 + a\sqrt{1 + y'^2} + b\,y \right) dx,$$

whence

$$y^2 + a\sqrt{1 + y'^2} + b y = \frac{a\, y'^2}{\sqrt{1 + y'^2}} + C;$$

$$\therefore \ y' \text{ or } \frac{dy}{dx} = \frac{\sqrt{\{a^2 - (y^2 + b\,y - C)^2\}}}{y^2 + b\,y - C},$$

or

$$dx = \frac{(y^2 + by - C)\, dy}{\sqrt{\{a^2 - (y^2 + by - C)^2\}}},$$

the equation to the elastic curve. Poisson's *Traité de Mécanique,* Tom. I. No. 310; Vince's *Fluxions,* Prob. CLXXII.

EXAMPLE IX.

Of all solids of revolution, and of equal mass, to determine that whose attraction upon a point situated in the axis is a maximum.

Let us take the given point as the origin of the co-ordinates x, y of the generating curve; then the attraction of a circular lamina of the body whose radius is y and thickness h upon the point is expressed by

$$2\pi\rho\left(1 - \frac{x}{\sqrt{x^2 + y^2}}\right)h,$$

ρ being a constant factor which expresses the intensity of the attractive force, referred to the unit of mass and of distance*.

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* For a clear explanation of the number ρ, vide Poisson's Traité de Méc. Tom. I. p. 170.

Hence the attraction of the entire solid, changing h into $d\,x$, will be

$$2\,\pi\,\rho \int_{x_0}^{x_1} \left(1 - \frac{x}{\sqrt{x^2+y^2}}\right) d\,x,$$

which is to be a maximum:

also,

$$t = \pi \int_{x_0}^{x_1} y^2\,d\,x = l.$$

$$\therefore\ \mathrm{U} = \int_{x_0}^{x_1} \left(1 - \frac{x}{\sqrt{x^2+y^2}} + a\,y^2\right) d\,x;$$

$$\mathrm{V} = 1 - \frac{x}{\sqrt{x^2+y^2}} + a\,y^2:$$

hence

$$N \text{ or } \frac{d\,\mathrm{V}}{d\,y} = 0,$$

becomes

$$\frac{x\,y}{\sqrt{(x^2+y^2)^3}} + 2\,a\,y = 0;$$

the equation to the generating curve.

EXAMPLE X.

Given the length (l) of a curve, to find its nature when the centre of gravity is most remote from the axis.

Let the axis of y be horizontal, that of x vertical; then the distance of the centre of gravity from the axis of y is expressed by

$$\frac{\int_{x_0}^{x_1} x\sqrt{1+y'^2}\,dx}{\int_{x_0}^{x_1}\sqrt{1+y'^2}\,dx} = \int_{x_0}^{x_1}\frac{x\sqrt{1+y'^2}}{l}\,dx.$$

Hence we have

$$U = \int_{x_0}^{x_1}\left(\frac{x\sqrt{1+y'^2}}{l} + a\sqrt{1+y'^2}\right)dx,$$

$$V = \left(\frac{x}{l} + a\right)\sqrt{1+y^2},$$

y'^2 /

and the general equation

$$N - \frac{d(P)}{dx} + \frac{d^2(Q)}{dx^2} - \&c. = 0,$$

becomes, by integration, $P = c$;

$$\therefore \frac{x}{l} + a = c\,\frac{\sqrt{1+y'^2}}{y'},$$

whence we deduce

$$dx = -\frac{c\,l\,dy'}{y'^2\sqrt{1+y'^2}};$$

$$\therefore \sqrt{1+y'^2}\,dx \text{ or } ds = -c\,l\,\frac{dy'}{y'^2},$$

and, integrating,

$$s = \frac{c\,l}{y'} + c', \text{ or } y' = \frac{c\,l}{s-c'}.$$

Now

$$\sqrt{1+y'^2} = \frac{ds}{dx},$$

$$\therefore dx = \frac{(s-c')\,ds}{\sqrt{\{(s-c')^2+c^2l^2\}}},\ dy = \frac{c\,l\,ds}{\sqrt{\{(s-c')^2+c^2l^2\}}},$$

equations to the catenary.

If the condition of equal length be omitted, that is, if $a=0$, we shall have $\frac{x}{l} = c\,\frac{\sqrt{1+y'^2}}{y'}$; and the nature of the resulting curve will be the same, s being represented by l; and if s be not represented by l, but be supposed to vary, the curve will still be a catenary. Vid. Sect. VII. Example II. p. 148.

EXAMPLE XI.

Of all Isochronous curves, to find *A B*, (fig. 7,) such that the space included between the curve *A B* and the chord drawn from *A* to *B* may be a maximum.

Here we have

$$v = \int_{x_0}^{x_1} \left(y\, d\, x - d . \frac{x\, y}{2} \right)$$

$$= \int_{x_0}^{x_1} \left(\frac{y}{2} - \frac{y'\, x}{2} \right) d\, x, \text{ a maximum };$$

and

$$t = \int_{x_0}^{x_1} \sqrt{\frac{1 + y'^2}{y}}\, d\, x = l :$$

l denoting a given time.

Hence,

$$\mathrm{U} = \int_{x_0}^{x_1} \left\{ \tfrac{1}{2} (y - y'\, x) + a \sqrt{\frac{1 + y'^2}{y}} \right\} d\, x ;$$

and

$$V = \tfrac{1}{2}(y - y'x) + a\sqrt{\frac{1+y'^2}{y}},$$

containing x, y and y'.

$$\therefore M = -\frac{y'}{2},\ P = -\frac{x}{2} + \frac{ay'}{\sqrt{y(1+y'^2)}};$$

and the equation,

$$V = \int M\,dx + Py' + C, \ (23),$$

becomes,

$$\tfrac{1}{2}(y - y'x) + a\sqrt{\frac{1+y'^2}{y}} = -\int\frac{y'\,dx}{2} - \frac{y'x}{2} + \frac{ay'^2}{\sqrt{y(1+y'^2)}} + C;$$

but

$$y' = \frac{dy}{dx}, \therefore \int\frac{y'\,dx}{2} = \frac{y}{2},$$

including the arbitrary constant in C;

hence, by reduction, we obtain,

$$y + \frac{a}{\sqrt{y(1+y'^2)}} = C,$$

and

$$y' \text{ or } \frac{dy}{dx} = \frac{\sqrt{\{a^2 - (C-y)^2 y\}}}{(C-y)\sqrt{y}}.$$

$$\therefore dx = \frac{(C-y)\sqrt{y}}{\sqrt{\{a^2-(C-y)^2 y\}}} dy,$$

which is the equation to the curve.

EXAMPLE XII.

Of all Isoperimetrical curves drawn between B and N, (fig. 8,) to find $B F N$ such that another curve $B Z N$ constructed upon the same base shall contain the greatest area, the ordinate $P Z$ being a given function of the ordinate $P F$.

Representing $P F$ by y, and $P Z$ (the given function of $P F$) by ψy, B being the origin of the coordinates, we have

$$v = \int_0^{x_1} \psi y \, dx, \text{ a maximum},$$

$$t = \int_0^{x_1} \sqrt{1+y'^2} \, dx = l;$$

l representing the given length.

$$\therefore U = \int_0^{x_1} \left(\psi y + a\sqrt{1+y'^2}\right) dx,$$

and

$$V = \psi y + a\sqrt{1+y'^2}.$$

The equation,

$$V = Py' + C$$

becomes

$$\psi y + a\sqrt{1+y'^2} = \frac{a y'^2}{\sqrt{1+y'^2}} + C,$$

or,

$$\frac{a}{\sqrt{1+y'^2}} = C - \psi y;$$

$$\therefore\ y' \text{ or } \frac{dy}{dx} = \frac{\sqrt{\{a^2-(C-\psi y)^2\}}}{C-\psi y},$$

and consequently,

$$dx = \frac{(C-\psi y)\,dy}{\sqrt{\{a^2-(C-\psi y)^2\}}}:$$

the differential equation to the curve, the integral of which will contain three arbitrary constants; to determine these we have given the two points B and N, and the length (l) of the curve.

SECTION VII.

31. WHEN V, as well as being a function of x, y, y', y'', &c., depends upon other integrals.

To find the variation of $\int \mathrm{V}\,dx$ in which V is a given function of x, y, y', y'', &c., and of $u = \int T\,dx$; T being also a given function of x, y, y', y'', &c.

Since,

$$\mathrm{V} = F\,(x, y, y', y'', \text{\&c.}, u),$$

$$\mathrm{T} = f\,(x, y, y', y'', \text{\&c.}),$$

$$d\mathrm{V} = M\,dx + N\,dy + P\,dy' + Q\,dy'' + \text{\&c.} + L\,du,$$

$$d\,\mathrm{T} = \mu\,dx + \nu\,dy + \pi\,dy' + \xi\,dy'' + \text{\&c.}$$

We have also from art. (16.) p. 42,

$$\delta \int \mathrm{V}\,dx = \mathrm{V}\,\delta x + \int (dx\,\delta \mathrm{V} - d\mathrm{V}\,\delta x),$$

which, by substituting for dV and δV these abbreviated values,

$$dV = d\psi + L\,du,$$
$$\delta V = \delta\psi + L\,\delta u$$

becomes

$$\delta\int V\,dx = V\,\delta x + \int(dx\,\delta\psi - d\psi\,\delta x)$$
$$+\int(L\,dx\,\delta u - L\,du\,\delta x).$$

Now observing that

$$du = T\,dx,$$
$$\delta u = \delta\int T\,dx = T\,\delta x + \int(dx\,\delta T - dT\,\delta x);$$

making $\int L\,dx = I$, and integrating by parts, the last term $\int(L\,dx\,\delta u - L\,du\,\delta x)$ becomes

$$I\int(dx\,\delta T - dT\,\delta x) - \int I\,(dx\,\delta T - dT\,\delta x);$$

substituting this in the preceding equation, we have

$$\left.\begin{array}{l}\delta\int V\,dx = V\,\delta x + \int(dx\,\delta\psi - d\psi\,\delta x)\\ \quad + I\int(dx\,\delta T - dT\,\delta x) - \int I\,(dx\,\delta T - dT\,\delta x)\end{array}\right\}\ldots(1)$$

the values of the integrals

$$\int (d\,x\,\delta\,\psi - d\,\psi\,\delta\,x), \quad \int (d\,x\,\delta\,T - d\,T\,\delta\,x),$$

will be obtained by the method followed in (16), since ψ and T are precisely similar to V in that article.

Again, the last term $\int I\,(d\,x\,\delta\,T - d\,T\,\delta\,x)$, by substituting for the quantity $d\,x\,\delta\,T - d\,T\,\delta\,x$, its value

$$\left(\nu\,\omega + \pi\,\frac{d\,\omega}{d\,x} + \xi\,\frac{d^2\,\omega}{d\,x^2} + \rho\,\frac{d^3\,\omega}{d\,x^3} + \&c.\right) d\,x,$$

will become

$$\int \left(I\,\nu\,\omega + I\,\pi\,\frac{d\,\omega}{d\,x} + I\,\xi\,\frac{d^2\,\omega}{d\,x^2} + I\,\rho\,\frac{d^3\,\omega}{d\,x^3} + \&c.\right) d\,x\,;$$

integrating by parts,

$$\int I\,\pi\,\frac{d\,\omega}{d\,x}\,d\,x = I\,\pi\,\omega - \int \frac{d\,(I\,\pi)}{d\,x}\,\omega\,d\,x,$$

$$\int I\,\xi\,\frac{d^2\,\omega}{d\,x^2}\,d\,x = I\,\xi\,\frac{d\,\omega}{d\,x} - \int \frac{d\,\omega}{d\,x}\,\frac{d\,(I\,\xi)}{d\,x}\,d\,x$$

$$= I\,\xi\,\frac{d\,\omega}{d\,x} - \frac{d\,(I\,\xi)}{d\,x}\,\omega + \int \frac{d^2\,(I\,\xi)}{d\,x^2}\,\omega\,d\,x\,;$$

&c.

Substituting these expressions in the value of $\int I\,(d\,x\;\delta\,T - d\,T\;\delta\,x)$, and afterwards replacing the three integrals of equation (1) by their indicated values, we shall have for the variation of $\int V\,d\,x$, in the case proposed,

$$\delta \int \mathrm{V}\,d\,x = \Gamma + \Delta + E$$

$$+ \int \mathrm{H}\,\omega\;d\,x + I \int \mathrm{K}\,\omega\,d\,x + \int I\;\Pi\;\omega\;d\,x; \ldots\ldots \text{(B)}$$

making for brevity

$$\mathrm{H} = N - \frac{d\,(P)}{d\,x} + \frac{d^2\,(Q)}{d\,x^2} - \text{\&c.}$$

$$\mathrm{K} = \nu - \frac{d\,\pi}{d\,x} + \frac{d^2\,(\xi)}{d\,x^2} - \text{\&c.}$$

$$\Pi = I\nu - \frac{d\,(I\,\pi)}{d\,x} + \frac{d^2\,(I\,\xi)}{d\,x^2} - \text{\&c.}$$

$$\Gamma = \mathrm{V}\;\delta\;x + \left(P - \frac{d\,(Q)}{d\,x} + \frac{d^2\,(R)}{d\,x^2} - \text{\&c.}\right)\omega$$

$$+ \left(Q - \frac{d\,(R)}{d\,x} + \text{\&c.}\right)\frac{d\,\omega}{d\,x}$$

$$+ \left(R - \text{\&c.}\right)\frac{d^2\,\omega}{d\,x^2}$$

$$+ \text{\&c.}$$

$$\begin{aligned}\Delta \quad = \quad & I\left(\pi - \frac{d\,(\xi)}{d\,x} + \frac{d^2\,(\rho)}{d\,x^2} - \&c.\right)\omega \\ & + I\left(\xi - \frac{d\,(\rho)}{d\,x} + \&c.\right)\frac{d\,\omega}{d\,x} \\ & + I\left(\rho - \&c.\right)\frac{d^2\,\omega}{d\,x^2} \\ & + \&c.\end{aligned}$$

$$\begin{aligned}E \quad = \quad & \left(I\pi - \frac{d(I\xi)}{d\,x} + \frac{d^2(I\rho)}{d\,x^2} - \&c.\right)\omega \\ & + \left(I\xi - \frac{d\,(I\rho)}{d\,x} + \&c.\right)\frac{d\,\omega}{d\,x} \\ & + \left(I\rho - \&c.\right)\frac{d^2\omega}{d\,x^2} \\ & + \&c.\end{aligned}$$

If we suppose

$$d\,V = d\,\psi + L\,d\,u + L_{,}\,d\,u_{,} + \&c.,$$

in which

$$u = \int T\,d\,x, \quad u_{,} = \int T_{,}\,d\,x, \ \&c.,$$

it is easy to see that $\delta \int V\,d\,x$ will become

$$V\,\delta x+\int(dx\,\delta\psi-d\psi\,\delta x)+\int(L\,dx\,\delta u-L\,du\,\delta x)$$
$$+\int(L_{,}dx\,\delta u-L_{,}du_{,}\delta x)\,;$$
$$+\ \&c.$$

and to obtain the expression for the variation of $\int V\,d\,x$, it is necessary to add to the preceding expression for $\delta\int V\,d\,x$ the developments of the terms

$$I_{,}\int(d\,x\,\delta\,T_{,}-d\,T_{,}\,\delta\,x),\ -\int I_{,}\,(d\,x\,\delta\,T_{,}-d\,T_{,}\,\delta\,x)$$

$I_{,}$ being $=\int L_{,}\,d\,x$: each new function $L_{(n)}\,du_{(n)}$, introducing into the expression for $\delta\int V\,d\,x$, the terms,

$$I_{(n)}\int(d\,x\,\delta\,T_{(n)}-d\,T_{(n)}\,\delta\,x),$$
$$-\int I_{(n)}\,(d\,x\,\delta\,T_{(n)}-d\,T_{(n)}\,\delta\,x),$$

which are similar to the preceding.

32. If the total variation of $\int$ V $d\,x$ be re-

quired from $x=x_0$ to $x=x_1$, x_0 and x_1 being constant quantities, the definite parts or the terms freed from the sign $\int$ in the expression for $\delta\int V\,dx$ will vanish; and making

$$\int_{x_0}^{x_1} L\,dx = \mathrm{I},$$

we shall have from the equation (B),

$$\delta\int_{x_0}^{x_1} \mathrm{V}\,dx = \int_{x_0}^{x_1}\left(N - \frac{d\,(P)}{dx} + \frac{d^2\,(Q)}{dx^2} - \&c.\right)\omega\,dx$$

$$+\int_{x_0}^{x_1}\left\{\left(\mathrm{I} - \int L\,dx\right)\pi - \frac{d\,(\mathrm{I} - \int L\,dx)\,\pi}{dx} + \&c.\right\}\omega\,dx,$$

since I is a constant quantity.

EXAMPLE I.

Of all Isoperimetrical curves drawn between the given points B and N, to find $B F N$ such that $B Z N$, constructed upon the same base, shall contain the greatest area, $P Z$ being a given function of the arc $B F$ (s). Fig. 8.

Here the maximum property,

$$v = \int_0^{x_1} \varphi s \, d x,$$

corresponds to

$$\int_{x_0}^{x_1} T \, d x;$$

and the isoperimetrical property being

$$t = \int_0^{x_1} \sqrt{1 + y'^2} \, d x,$$

we have by the principle of (28), p. 104.

$$\mathrm{V} = a \sqrt{1 + y'^2} + T,$$

and

$$d \mathrm{V} = \frac{a y'}{\sqrt{1 + y'^2}} d y' + \frac{d T}{d s} d s;$$

which, compared with

$$d\,\mathrm{V} = M\,dx + N\,dy + P\,dy' + \&\text{c.} + L\,du,$$

gives

$$M=0,\ N=0,\ P=\frac{a y'}{\sqrt{1+y'^2}},\ L=\frac{dT}{ds},\ ds=du.$$

$$\therefore\ \sqrt{1+y'^2}=T \text{ and } \pi=\frac{y'}{\sqrt{1+y'^2}}:$$

hence the formula of solution, art. (32), p. 142

$$\delta\int_0^{x_1} V\,dx = \int_0^{x_1}\left(N-\frac{d\,(P)}{dx}+\&\text{c.}\right)\omega\,dx$$

$$+\int_0^{x_1}\left\{(\mathrm{I}-\textstyle\int L\,dx)\nu-\frac{d\,(\mathrm{I}-\int L\,dx)\pi}{dx}+\&\text{c.}\right\}\omega\,dx=0,$$

gives

$$a\,d\cdot\frac{y'}{\sqrt{1+y'^2}}+d\cdot\frac{(\mathrm{I}-\int L\,dx)\,y'}{\sqrt{1+y'^2}}=0:$$

integrating, we have

$$a y' + \left(\mathrm{I} - \int L\, d x\right) y' = C \sqrt{1+y'^2},$$

or

$$\mathrm{I} - \int L\, d x = \frac{C \sqrt{1+y'^2}}{y'} - a;$$

$$\therefore d\left(\mathrm{I} - \int L\, d x\right) = \frac{C\, d y'}{\sqrt{1+y'^2}} - \frac{C\, d y' \sqrt{1+y'^2}}{y'^2},$$

$$= - \frac{C\, d y'}{y'^2 \sqrt{1+y'^2}},$$

whence

$$L\, d x = \frac{C\, d y'}{y'^2 \sqrt{1+y'^2}};$$

$$\therefore L\, d s = \frac{C\, d y'}{y'^2},$$

integrating,

$$T = \frac{C}{y'} + C',$$

from which we have

$$d x = \frac{d y\, (T - C')}{C};$$

the differential equation to the curve.

Hence,

$$ds = \sqrt{\left\{1 + \left(\frac{T - C'}{C}\right)^2\right\}}\, dy;$$

$$dy = \frac{C\, ds}{\sqrt{\left\{C^2 - (\phi s - C')^2\right\}}},\ dx = \frac{(\phi s - C')\, ds}{\sqrt{\left\{C^2 + (\phi s - C')^2\right\}}}.$$

The last two examples compose the celebrated isoperimetrical problem proposed by James Bernoulli to his brother John at the end of his solution to the problem of the curve of quickest descent. For an interesting history of this problem, perhaps the most difficult that was ever proposed, vide Woodhouse's *Isoperimetrical Problems*, p. 14, or Bossut's *General History of Mathematics*, p. 335.

EXAMPLE II.

Suppose a chain or cord of variable thickness to be attached to two fixed points; it is required to find the nature of the curve which it ought to form, so that its centre of gravity shall be the lowest possible.

Let the weight of an element of the curve be expressed by $\theta\, d\, s$, θ being a function of s depending upon the law of the thickness of the chain; then the distance of the centre of gravity from the horizontal axis of y will be expressed by

$$\frac{\int_{x_0}^{x_1} x\,\theta\, d s}{\int_{x_0}^{x_1} \theta\, d s},$$

which is to be a maximum.

Hence we have

$$\delta\frac{\int_{x_0}^{x_1} x\,\theta\,d\,s}{\int_{x_0}^{x_1}\theta\,d\,s}=\frac{\int_{x_0}^{x_1}\theta\,ds\,.\,\delta\int_{x_0}^{x_1}x\,\theta\,d\,s-\int_{x_0}^{x_1}x\,\theta\,ds\,.\,\delta\int_{x_0}^{x_1}\theta\,ds}{\left(\int_{x_0}^{x_1}\theta\,d\,s\right)^2}.$$

$$=0\,;$$

$$\therefore\int_{x_0}^{x_1}\theta\,d\,s\,.\,\delta\int_{x_0}^{x_1}x\,\theta\,ds-\int_{x_0}^{x_1}x\,\theta\,ds\,.\,\delta\int_{x_0}^{x_1}\theta\,d\,s=0\ldots(1)$$

the terms of which we must develop separately.

Since,

1. $$\delta\int_{x_0}^{x_1}x\,\theta\,d\,s=\delta\int_{x_0}^{x_1}x\,\theta\sqrt{1+y'^2}\,d\,x,$$

$$\mathrm{V}=x\,\theta\sqrt{1+y'^2},$$

and

$$d\,V=\theta\sqrt{1+y'^2}\,d\,x+\frac{x\,\theta\,y'\,d\,y'}{\sqrt{1+y'^2}}+x\sqrt{1+y'}\,,\frac{d\,\theta}{d\,s}\,d\,s,$$

142, therefore the equation in art. (32), becomes,

$$\delta\int_{x_0}^{x_1}x\,\theta\,d\,s=-\int_{x_0}^{x_1}\left\{\frac{d\left(\frac{x\,\theta\,y'}{\sqrt{1+y'^2}}\right)}{d\,x}+\frac{d\left(\frac{\mathrm{I}-\int(L\,dx)y'}{\sqrt{1+y'^2}}\right)}{d\,x}\right\}\omega\,dx,$$

since

$$N=0,\ P=\frac{x\,\theta\,y'}{\sqrt{1+y'^2}},\ Q=0,\ \&c.;$$

$$\nu=0,\ \pi=\frac{y'}{\sqrt{1+y'^2}},\ \xi=0,\ \&c.$$

But

$$\int L\,dx=x\sqrt{1+y'^2}\,\frac{d\theta}{ds}\,dx=\int x\,d\theta=x\theta-\int\theta\,dx;$$

hence the preceding equation by substitution and reduction becomes

$$\delta\int_{x_0}^{x_1}x\,\theta\,ds$$

$$=-\int_{x_0}^{x_1}\left\{\frac{d\left(\dfrac{\mathrm{I}\,y'+y'\int\theta\,dx}{\sqrt{1+y'^2}}\right)}{dx}\right\}\omega\,dx$$

in which,

$$\mathrm{I}\text{ or}\int_{x_0}^{x_1}L\,dx=x_1\,\theta_1-x_0\,\theta_0-\int_{x_0}^{x_1}\theta\,dx.$$

2. Again,

$$\delta\int_{x_0}^{x_1}\theta\,ds=\delta\int_{x_0}^{x_1}\theta\sqrt{1+y'^2}\,dx;$$

$$\therefore\ V=\theta\sqrt{1+y'^2},$$

$$dV=\frac{\theta y'\,dy'}{\sqrt{1+y'^2}}+\sqrt{1+y'^2}\,\frac{d\theta}{ds}\,ds,$$

and the equation referred to above, in this case, becomes

$$\delta\int_{x_0}^{x_1}\theta\,ds$$

$$=-\int_{x_0}^{x_1}\left\{\frac{d\left(\frac{\theta y'}{\sqrt{1+y'^2}}\right)}{dx}+\frac{d\left(\frac{(\mathrm{I}'-\theta)\,y'}{\sqrt{1+y'}}\right)}{dx}\right\}\omega\,dx\,;$$

N being $=0$, $P=\frac{\theta y'}{\sqrt{1+y'^2}}$, $\int L\,dx$, or

$\int\sqrt{1+y'^2}\,\frac{d\theta}{ds}\,dx=\theta$; and I′, being analogous to I, represents the entire value of θ between the values of $x=x_0$ and $x=x_1$, that is, $\theta_1-\theta_0$.

Hence we have

$$\delta\int_{x_0}^{x_1}\theta\,ds=-\int_{x_0}^{x_1}\frac{d\left(\frac{\mathrm{I}'\,y'}{\sqrt{1+y'^2}}\right)}{dx}\,\omega\,dx.$$

Substituting in the equation (1), the values of $\delta\int_{x_0}^{x_1}\theta\, x\, d\, s$ and $\delta\int_{x_0}^{x_1}\theta\, d\, s$ given above, and representing by A and B the definite integrals $\int_{x_0}^{x_1} x\,\theta\, d\, s$ and $\int_{x_0}^{x_1}\theta\, d\, s$, we obtain

$$\int_{x_0}^{x_1}\left\{A\frac{d\left(\frac{\mathrm{I}'\, y'}{\sqrt{1+y'^2}}\right)}{d\, x}-B\frac{d\left(\frac{\mathrm{I}\, y'+y'\int\theta\, d\, x}{\sqrt{1+y'^2}}\right)}{d\, x}\right\}\omega\, dx=0,$$

whence we have, (20),

$$\lambda\, d\, .\frac{\mathrm{I}'\, y'}{\sqrt{1+y'^2}}-d\, .\frac{\mathrm{I}\, y'+y'\int\theta\, d\, x}{\sqrt{1+y'^2}}=0,$$

λ being put for $\frac{A}{B}$; and, integrating,

$$\mathrm{I}\, y'+y'\int\theta\, d\, x-\lambda\,\mathrm{I}'\, y'+C\sqrt{1+y'^2}=0,$$

C being an arbitrary constant.

Now I, I′, and λ being constant quantities, we shall have, by differentiation,

$$\theta\, d x = - \frac{C\, d y'}{y'^2 \sqrt{1+y'^2}};$$

whence,

$$\theta\, d x \sqrt{1+y'^2} = - \frac{C\, d y'}{y'^2},$$

integrating,

$$\int \theta\, d s = \frac{C}{y'} + C',$$

C' being another arbitrary constant.

We have therefore,

$$y' \text{ or } \frac{d y}{d x} = \frac{C}{\int \theta\, d s - C'};$$

and since

$$\sqrt{1+y'^2} = \frac{d s}{d x},$$

$$d x = \frac{\int \theta\, d s - C'}{\sqrt{\{(\int \theta\, d s - C')^2 + C^2\}}} d s,$$

and

$$d y = \frac{C\, d s}{\sqrt{\{(\int \theta\, d s - C')^2 + C^2\}}}.$$

If the chain be uniformly thick, θ is constant, and

$$d\,x = \frac{(m\,s - C')\,d\,s}{\sqrt{\{(ms - C')^2 + C^2\}}},$$

$$d\,y = \frac{C\,d\,s}{\sqrt{\{(m\,s - C')^2 + C^2\}}};$$

which are the same equations as those of Example X. Sect. VI., and belong to the catenary.

33. Suppose T to contain a new integral $u_{,} = \int T_{,}\,d\,x$, or to find the variation of $\int \mathrm{V}\,d\,x$ in which V is a given function of x, y, y', y'', &c., and of $u = \int T\,d\,x$; T being also a given function of x, y, y', y'', &c., and of $u_{,} = \int T_{,}\,d\,x$; and $T_{,}$, a given function of x, y, y', y'', &c.

In this case,

$$d\,T = \mu\,d\,x + \nu\,d\,y + \pi\,d\,y' + \xi\,d\,y'' + \&c. + L_{,}\,d\,u,$$
$$d\,T_{,} = \mu_{,}\,d\,x = \nu_{,}\,d\,y + \pi_{,}\,d\,y' + \xi_{,}\,d\,y'' + \&c.;$$

and, making as before,

$$d\,\mathrm{V} = d\,\psi + L\,d\,u,$$

we have

$$\delta \int \mathrm{V}\, dx = \mathrm{V}\, \delta x + \int (dx\, \delta\psi - d\psi\, \delta x) + \int (L\, dx\, \delta u - L\, du\, \delta x).$$

Also, since

$$\int (L\, dx\, \delta u - L\, du\, \delta x) = \int \{L\, dx \int (dx\, \delta T - dT\, \delta x)\},$$

if we replace $d\,T$ and $\delta\,T$ by $d\,\psi_{,} + L_{,}\,d\,u_{,}$ and $\delta\,\psi_{,} + L_{,}\,\delta\,u_{,}$, we shall have

$$\int (L\, dx\, \delta u - L\, du\, \delta x) = \int \{L\, dx \int (dx\, \delta\psi_{,} - d\psi_{,}\, \delta x)\}$$
$$+ \int \{L\, dx \int (L_{,}\, dx\, \delta u_{,} - L_{,}\, du_{,}\, \delta x)\}$$

or, putting in the place of $du_{,}$ and $\delta\,u_{,}$, their values $T_{,}\,dx$ and $T_{,}\,\delta x$, we shall have

$$\int (L\, dx\, \delta u - L\, du\, \delta x) = \int \{L\, dx \int (dx\, \delta\psi_{,} - d\psi_{,}\, \delta x)\}$$
$$+ \int \left\{ L\, dx \int \{L_{,}\, dx \int (dx\, \delta T_{,} - dT_{,}\, \delta x)\} \right\};$$

whence we have

$$\delta \int \mathrm{V}\, dx = \mathrm{V}\, dx + \int (dx\, \delta\psi - d\psi\, \delta x) + \int \{L\, dx \int (dx\, \delta\psi_{,} - d\psi_{,}\, \delta x)\}$$
$$+ \int \left\{ L\, dx \int \{L_{,}\, d\,x \int (dx\, \delta\, T_{,} - d\,T_{,}\, \delta x)\} \right\}.$$

Now $\psi_{,}$ is the same as T in art. (31), hence it is only necessary to find the value of the last term,

$$\int \left\{ L\, dx \int \{L_{,}\, dx \int (dx\, \delta T_{,} - d\,T_{,}\, \delta\, x)\} \right\},$$

and to add it to the development of $\delta\int V\,dx$, given in the preceding article, to have the variation sought.

Let us make $\int L\,dx = I$, and integrating by parts, we shall at first have

$$I\int\{L_{\prime}dx\int(dx\,\delta T_{\prime}-dT_{\prime}\,\delta x)\}$$

$$-\int\{I L\,dx\int(dx\,\delta T_{\prime}-dT_{\prime}\,\delta x)\};$$

again, making $\int L_{\prime}\,dx = I_{\prime}$, and $\int I L\,dx = K_{\prime}$, we shall find for the first of these terms,

$$I I_{\prime}\int(dx\,\delta T_{\prime}-dT_{\prime}\,\delta x) - I\int I_{\prime}(dx\,\delta T_{\prime}-dT_{\prime}\,\delta x);$$

and for the second

$$-K_{\prime}\int(dx\,\delta T_{\prime}-dT_{\prime}\,\delta x) + \int K_{\prime}(dx\,\delta T_{\prime}-dT_{\prime}\,\delta x),$$

Of the four new terms just obtained, two are similar to $I\int(dx\,\delta T - dT\,\delta x)$, and the two others, to $\int I\,(dx\,\delta T - d\,T\,\delta\,x)$, of which we have treated before (31). Hence the complete development of the variation of $\int V\,dx$ in this case will be obtained.

From what is done above the method of proceeding will be evident if $T_{,}$ should contain an integral $u_{,,} = \int T_{,,} \, dx$.

Generally, if

$$
\begin{aligned}
V &= F\,(x, y, y', y'', \text{\&c.}, u), \\
u &= \int T \, dx\,; \\[1ex]
T &= F_1\,(x, y, y', y'', \text{\&c.}, u_{,}), \\
u_{,} &= \int T_{,} \, dx\,; \\[1ex]
T_{,} &= F_2\,(x, y, y', y'', \text{\&c.}, u_{,,}), \\
u_{,,} &= \int T_{,,} \, dx; \\[1ex]
&\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\[1ex]
T_{(n-1)} &= F_n\,(x, y, y'\ y'', \text{\&c.}, u_{(n)}), \\
u_{(n)} &= T_{(n)} \, dx\,;
\end{aligned}
$$

the variation of $\int V\,dx$ will be obtained without any other difficulty than what may arise from the length of the calculations.

34. If V, T, $T_{,}$, &c., be functions of three variables x, y, z, we may, as was done in (22), extend the preceding formulæ to comprehend this case. p. 54

In fact, let

$$
\begin{aligned}
d\,\mathrm{V} = M\,dx + N\,dy + P\,dy' + Q\,dy'' + R\,dy''' + \text{\&c.} \\
+ N_{,}dz + P_{,}\,dz' + Q_{,}\,dz'' + R_{,}d\,z''' + \text{\&c.} \\
+ L\,du\,;
\end{aligned}
$$

in which

$$y' = \frac{d\,y}{d\,x},\ y'' = \frac{d^2\,y}{d\,x^2},\ \text{\&c.},\ z' = \frac{d\,z}{d\,x},\ z'' = \frac{d^2\,z}{d\,x^2},\ \text{\&c.},$$

$d\,u = T\,d\,x.$

$$\begin{aligned} d\,T = \mu\,d\,x + \nu\,d\,y + \pi\,d\,y' + \xi\,d\,y'' + \rho\,d\,y''' + \text{\&c.} \\ + \nu_{,} d\,z_{,} + \pi_{,}\,d\,z' + \xi_{,}\,d\,z'' + \rho_{,}\,d\,z''' + \text{\&c.} \\ + L_{,}\,du_{,} \end{aligned}$$

$$d\,u_{,} = T_{,}\,d\,x,$$

it is at first evident from the preceding article that we shall have,

$$\delta \int \mathrm{V}\,d\,x = \mathrm{V}\,\delta\,x + \int (d\,x\,\delta\,\psi - d\,\psi\,\delta\,x)$$

$$+ \int \left\{ L\,d\,x \int (dx\,\delta\psi_{,} - d\psi_{,}\,\delta x) \right\} + \int \left\{ L\,d\,x \int \left\{ L_{,} dx \int (dx\,\delta T_{,} - dT_{,}\delta x) \right\} \right\}$$

and thus we arrive at the same reductions; we have then only to develop the quantities of the form $d\,x\ \delta\,\psi - d\,\psi\ \delta\,x$. By art. (16) we have

$$\begin{aligned} dx\,\delta\psi - d\psi\,\delta x = N\,dx\,(\delta y - y'\,\delta x) + P\,dx\,(\delta y' - y''\,\delta x) + \text{\&c.} \\ + N_{,}\,dx\,(\delta z - z'\,\delta x) + P_{,}\,dx\,(\delta z' - z''\,\delta x) + \text{\&c.}\,; \end{aligned}$$

making for brevity,

$$\delta\,y - y'\,\delta\,x = \omega, \qquad d\,z - z'\,\delta x = \omega_{,},$$

we shall have

$$dx\,\delta\psi - d\psi\,\delta x = (N\,\omega + P\,\omega' + Q\,\omega'' + \&c.)\,d\,x$$
$$+ (N_{,}\,\omega_{,} + P_{,}\,\omega'' + Q_{,}\,\omega''_{,} + \&c.)\,d\,x:$$

and performing the integrations by parts, we find two series of terms, the one in ω, N, P, Q, &c., and the other in $\omega_{,}$, $N_{,}$, $P_{,}$, $Q_{,}$, &c., exactly similar to those to which we have before so frequently referred.

SECTION VIII.

35. To find the variation of a function which is only given by means of a differential equation.

Let the function be $\int u$, u depending upon a differential equation involving no differential of u higher than the first order: let us suppose this equation to be

$$du - X\,dx + V\,dx = 0,$$

X being a given function of x, y, y', y'', &c., V a given function of x, y, y', y'', &c., and u.

Taking its differential, (dx being constant,) we have

$$d^2u - \left(\frac{dX}{dx}dx + \frac{dX}{dy}\cdot\frac{dy}{dx}dx + \frac{dX}{dy'}\cdot\frac{dy'}{dx}dx + \text{\&c.}\right)dx$$

$$+\left(\frac{dV}{dx}dx + \frac{dV}{dy}\cdot\frac{dy}{dx}dx + \frac{dV}{dy'}\cdot\frac{dy'}{dx}dx + \text{\&c.}\right)dx$$

$$+\frac{dV}{du}\,du\,dx = 0.$$

Representing the part of the given equation which is independent of u, by the symbol ϕ, and $\frac{dV}{du}$ by T, we may write the preceding equation thus,

$$d^2 u - d\,\phi\, d\,x + T\, d\,u\; d\,x = 0.$$

Changing a differential into a variation, or, which is the same thing, taking the variation of the given equation we have

$$d\,\delta\, u - \delta\,\phi\, dx + T\,\delta\, u\; d\,x = 0.$$

Now to deduce the value of $\delta \int u$ or $\int \delta\, u$, we shall multiply the last equation by the indeterminate factor λ, and we have

$$\lambda\, d\,\delta\, u - \lambda\;\delta\,\phi\; d\,x + \lambda\; T\,\delta\, u\; dx = 0,$$

or, under another form,

$$d\,.\,\lambda\;\delta u - \delta\, u\; d\,\lambda - \lambda\;\delta\,\phi\; d\,x + \lambda\; T\,\delta\, u\; d\,x = 0 \ldots (1)$$

This equation being of the first degree and first order, and having introduced an arbitrary factor λ, it is manifest that we may assume for the determination of this factor,

$$\lambda\; T\;\delta\; u\; d\,x - \delta\; u\; d\,\lambda = 0,$$

provided we substitute in the sum of the remaining terms equated to zero, the value thus obtained: whence we have

$$\frac{d\,\lambda}{\lambda} = T\,d\,x,$$

and integrating,

$$\log \lambda = \int T\,d\,x,$$

or

$$\lambda = e^{\int T\,d\,x}.$$

Substituting this in the sum of the first and third terms of equation (1), we shall have

$$d\,.\,e^{\int T\,dx}\,\delta u = e^{\int T\,dx}\,\delta\,\phi\,d\,x,$$

or,

$$e^{\int T\,dx}\,\delta\,u = \int e^{\int T\,dx}\,\delta\,\phi\,dx;$$

$$\therefore\ \delta u = e^{-\int T\,dx} \int e^{\int T\,dx}\,\delta\,\phi\,d\,x;$$

consequently,

$$\int \delta\,u \text{ or } \delta \int u = \int e^{-\int T\,dx} \int e^{\int T\,dx}\,\delta\,\phi\,d\,x:$$

where it is unnecessary to add any constant, since we may suppose it comprised in the indicated integrations.

We may develop the factor $\int e^{\int T\,dx}\,\delta\varphi\,dx$ by the formula in art. (14), calling $e^{\int T\,dx}$, I; and substituting for M, N, P, &c., m, n, p, &c., in that article, the quantities

p. 33

$$I\left(-\frac{dX}{dx}+\frac{dV}{dx}\right), I\left(-\frac{dX}{dy}+\frac{dV}{dy}\right), I\left(-\frac{dX}{dy'}+\frac{dV}{dy'}\right),$$

which correspond to them.

36. In general let the differential equation be of any order whatever, and contain any number of variables; then the equation being of this form,

$$T\,du+T_1\,d^2u+T_2\,d^3u+\&c.+d\varphi=0,$$

φ being the part of the primitive equation which is independent of u, we may, by the substitution of the symbol δ for d, give it the following form,

$$T\delta u+T_1\,\delta\, du+T_2\,\delta\, d^2 u+\&c.+\delta\varphi=0,$$

or

$$T\,\delta u+T_1\, d\,\delta u+T_2\, d^2\,\delta u+\&c.+\delta\varphi=0\,;$$

an equation of the first degree with respect to δu. Hence multiplying by the indeterminate variable factor λ, and affecting each term with the sign $\int$, we shall have

$$\int\lambda T\delta u+\int\lambda T_1 d\,\delta u+\int\lambda T_2 d^2\delta u+\&c.+\int\lambda\delta\varphi=0.$$

Integrating by parts,

$$\int\lambda\; T_1\, d\,\delta u=\lambda T_1\,\delta u-\int\delta u\, d(\lambda\, T_1),$$

$$\int\lambda\, T_2\, d^2\,\delta u=\lambda\, T_2\, d\,\delta u-\delta u\, d\,(\lambda\, T_2)$$
$$+\int\delta u\, d^2\,(\lambda\, T_2),$$

$$\int\lambda\; T_3\, d^3\,\delta u=\lambda\; T_3\, d^2\,\delta u-d\,\delta u\, d\,(\lambda\, T_3)$$
$$+\delta u\, d^2\,(\lambda\, T_3)-\int\delta u\, d^3\,(\lambda T_3),$$

&c.

Hence the preceding equation becomes

$$\left.\begin{array}{l}(\lambda\,T_1 - d\,(\lambda\,T_2) + d^2\,(\lambda\,T_3) - \&c.)\,\delta\,u \\ +\,(\lambda\,T_2 - d\,(\lambda\,T_3) + \&c.)\,d\,\delta\,u \\ +\,(\lambda\,T_3 - \&c.)\,d^2\,\delta\,u \\ +\ \ \&c. \\ +\int\lambda\,\delta\,\phi \\ +\int\left\{\lambda T - d(\lambda T_1) + d^2(\lambda\,T_2) - d^3(\lambda T_3) + \&c.\right\}\delta\,u\end{array}\right\} = 0.$$

Assuming the coefficient of $\delta\,u$ under the integral sign equal to zero, we shall have

$$\lambda\,T - d\,(\lambda\,T) + d^2\,(\lambda\,T_2) - d^3(\lambda\,T_3) + \&c. = 0\,;$$

the complete integral of which will contain as many arbitrary constants as are equal to the number (n) of its order; and, consequently, equal to that of the variations $\delta\,u,\ d\,\delta\,u,\ d^2\,\delta\,u \ldots d^{n-1}\delta\,u$, in the part of the preceding equation freed from the sign $\int$. These constants will enable us to equate to zero each of the coefficients of $d\,\delta\,u,\ d^2\,\delta\,u$, &c., in that part, when we shall have assigned to x, y, &c., the values at the limits.

In general, from the equation

$$\lambda\,T - d\,(\lambda\,T_1) + d^2(\lambda\,T_2) - \&c. = 0,$$

we may determine the factor λ, the value of which we shall have to substitute in the equation

$$\left.\begin{aligned}&(\lambda T_1-d(\lambda T_2)+d^2(\lambda T_3)-\&c.)\,\delta u\\&+(\lambda T_2-d(\lambda T_3)+\&c.)\,d\,\delta u\\&+(\lambda T_3-\&c.)\,d^2\,\delta u\\&+\ \&c.\\&+\int\lambda\,d\varphi\end{aligned}\right\}=0$$

37. If we suppose the variables x, y, z, &c. to be subject to the condition expressed by the equation $\Phi=0$, so that

$$X\,dx+Y\,dy+Z\,dz+\&c.=0,$$
$$X\,\delta x+Y\,\delta y+Z\,\delta z+\&c.=0;$$

we shall have the expression of one of the variations in a function of the others, which we may substitute in the value of δu, found by the preceding formulas, and the number of variations will be diminished by unity in the result.

The same method may be used when more conditions are to be fulfilled, by combining the equations which connect the variables with those relative to other conditions of the problem.

38. If the differential equation be represented by

$$V=0,$$

V being a given function of x, y, y', y'', &c., u, u', u'', &c., in which $y' = \frac{dy}{dx}$, $y'' = \frac{d^2y}{dx^2}$, &c., $u' = \frac{du}{dx}$, $u'' = \frac{d^2u}{dx^2}$, &c., we must determine the development of $\delta V = 0$; but since $dV = 0$, we shall evidently have the equation,

$$\int (dx\,\delta V - dV\,\delta x) = 0,$$

which is of the same form as that in (16). p. 42

Let

$$dV = M\,dx + N\,dy + P\,dy' + Q\,dy'' + \&c. \\ + T\,du + T_1\,du' + T_2\,du'' + \&c.,$$

$$\delta V = M\,\delta x + N\,\delta y + P\,\delta y' + Q\,\delta y'' + \&c. \\ + T\,\delta u + T_1\,\delta u' + T_2\,\delta u'' + \&c.;$$

and if we make

$$\delta y - y'\,\delta x = \omega,$$
$$\delta u - u'\,\delta x = \Omega,$$

we shall have

$$dx\,\delta V - dV\,\delta x = (N\omega + P\omega' + Q\omega'' + \&c.) \\ + (T\Omega + T_1\Omega' + T_2\Omega'' + \&c.)\,dx.$$

Now if we multiply by λ, affect each term with the sign $\int$, and integrate by parts, following the process in the article above mentioned; the equation $\int \lambda\,(d\,x\;\delta\,V - d\,V\;\delta\,x) = 0$ will become

$$\Gamma + \mathrm{E} + \int \mathrm{H}\,\omega\,d\,x + \int \mathrm{K}\,\Omega\,d\,x = 0;$$

making, for the sake of brevity,

$$\mathrm{H} = \lambda\,N - \frac{d\,(\lambda\,P)}{d\,x} + \frac{d^2\,(\lambda\,Q)}{d\,x^2} - \&c.$$

$$\mathrm{K} = \lambda\,T - \frac{d\,(\lambda\,T_1)}{d\,x} + \frac{d^2\,(\lambda\,T_2)}{d\,x^2} - \&c.$$

$$\Gamma = \left(\lambda\,P - \frac{d\,(\lambda\,Q)}{d\,x} + \frac{d^2\,(\lambda\,R)}{d\,x^2} - \&c.\right)\omega$$

$$+\left(\lambda\,Q - \frac{d\,(\lambda\,R)}{d\,x} + \&c.\right)\omega'$$

$$+\ (\lambda\,R - \&c.)\ \omega''$$

$$+\ \&c.$$

$$\mathrm{E} = \left(\lambda T_1 - \frac{d\,(\lambda T_2)}{d\,x} + \frac{d^2\,(\lambda\,T_3)}{d\,x^2} - \&c.\right)\Omega$$

$$+\left(\lambda\,T_2 - \frac{d\,(\lambda\,T_3)}{d\,x} + \&c.\right)\Omega'$$

$$+(\lambda\,T_3 - \&c.)\ \Omega''$$

$$+\ \&c.$$

When u is a maximum or minimum, $\Omega=0$, and the definite expression E vanishes. If we assume for the determination of λ,

$$K=0;$$

we shall have

$$\Gamma+\int H\,\omega\,dx=0,$$

of which the part within the sign $\int$ furnishes the equation

$$H=0.$$

39. The equation $V=0$ may be called the equation of condition; and when the variables are further required to make the function $\int V_{\prime}\,dx$ a maximum or minimum; we may, as in relative maxima
104 and minima (28), multiply V by the indeterminate factor λ, and take the variation of $U=\int\lambda V\,dx+\int V_{\prime}\,dx$. Hence we shall have,

$$\delta U=\delta\int(\lambda V+V_{\prime})\,dx=0.$$

Let

$$\begin{aligned} dV=&M\,dx+N\,dy+P\,dy'+Q\,dy''+\&c.\\ &+T\,du+T_1\,du'+T_2\,du''+\&c.\end{aligned}$$

$$\begin{aligned} dV_{\prime}=&M_{\prime}\,dx+N_{\prime}\,dy+P_{\prime}\,dy'+Q_{\prime}\,dy''+\&c.\\ &+\nu\,du+\pi\,du'+\xi\,du''+\&c.\end{aligned}$$

then if, for brevity, we make

$$\mathrm{H}=\lambda N-\frac{d(\lambda P)}{dx}+\frac{d^2(\lambda Q)}{dx^2}-\&c.+N_{,}-\frac{d(P_{,})}{dx}+\frac{d^2(Q_{,})}{dx^2}-\&c.$$

$$\mathrm{K}=\lambda T-\frac{d(\lambda T_1)}{dx}+\frac{d^2(\lambda T_2)}{dx^2}-\&c.+\pi-\frac{d(\pi)}{dx}+\frac{d^2(\xi)}{dx^2}-\&c.$$

$$\begin{aligned}\Gamma=&(\lambda V+V_{,})\,\delta x\\ &+\left(\lambda P-\frac{d(\lambda Q)}{dx}+\&c.+P_{,}-\frac{d(Q_{,})}{dx}+\&c.\right)\omega\\ &+(\lambda Q-\&c.+Q_{,}-\&c.)\omega'\\ &+\&c.\end{aligned}$$

$$\begin{aligned}\mathrm{E}=&\left(\lambda T_1-\frac{d(\lambda T_2)}{dx}+\&c.+\pi-\frac{d(\xi)}{dx}+\&c.\right)\Omega\\ &+(\lambda T_2-\&c.+\xi-\&c.)\,\Omega'\\ &+\&c.\end{aligned}$$

the equation $\delta\,\mathrm{U}=0$ will become

$$\Gamma+\mathrm{E}+\int\mathrm{H}\,\omega\,dx+\int\mathrm{K}\,\Omega\,dx=0,$$

in which we may treat the variations of y and u as independent, in consequence of the indeterminate quantity λ. Hence the preceding equation may be satisfied generally by decomposing it into the three equations

$$\Delta = 0,$$
$$H=0,\ K=0,$$

writing Δ for the definite part $\Gamma + E$.

From the latter equations, λ is to be eliminated; and the resulting equation, combined with the equation of condition $V=0$, will determine the functions y and u in terms of x and constant quantities.

And from the equation $\Delta=0$, or

$$\left.\begin{array}{l}
(\lambda V + V_{,})\,\delta x \\
+\left(\lambda P - \dfrac{d(\lambda Q)}{dx} + \&c. + P_{,} - \dfrac{d(Q_{,})}{dx} + \&c.\right)\omega \\
+(\lambda Q - \&c. + Q_{,} - \&c.)\,\omega' \\
+\ \&c. \\
+\left(\lambda T_1 - \dfrac{d(\lambda T_2)}{dx} + \&c. + \pi - \dfrac{d(\xi)}{dx} + \&c.\right)\Omega \\
+(\lambda T_2 - \&c. + \xi - \&c.)\,\Omega' \\
+\ \&c.
\end{array}\right\} = 0,$$

we may deduce the equation of limits,

$$\Delta_1 - \Delta_0 = 0.$$

The following examples will illustrate the preceding articles.

EXAMPLE I.

Required the curve which a body must describe, descending in a resisting medium, to acquire the greatest velocity.

Let v represent the velocity, g the constant force of gravity, $a\psi v$ the resistance, ψv being a given function of the resistance, and a a constant depending upon the intensity or measure of the resistance; also let x be supposed vertical, y horizontal, and ds an element of the curve.

The accelerating forces which act upon the body, arising from gravity and the resistance, will be expressed by

$$g\frac{dx}{ds}, -a\psi v;$$

hence by the principles of Mechanics, we have

$$v\,dv=\left(g\frac{dx}{ds}-a\psi v\right)ds,$$

or

$$d\,.\,v^2-2g\,dx+2a\,\psi\,v\sqrt{1+y'^2}\,dx=0:$$

put u for v^2, and $\phi\,u$ for $\psi\,v$; then this equation may be written

$$du-2g\,dx+2a\,\phi\,u\sqrt{1+y'^2}\,dx=0:$$

$$\therefore V \text{ or } \frac{du}{dx}-2g+2a\,\phi\,u\sqrt{1+y'^2}=0.$$

Differentiating, we have

$$\frac{d^2u}{dx}+2a\frac{d\,.\,\phi\,u}{du}\sqrt{1+y'^2}\,du+\frac{2a\,\phi\,u\,y'\,dy'}{\sqrt{1+y'^2}}=0.$$

Comparing this with the value of $d\,V$ in (38), p. 166 we have

$$M=0,\ N=0,\ P=\frac{2a\,\phi\,u\,y'}{\sqrt{1+y'^2}},\ Q=0,\ \&c.,$$

$$T=2a\frac{d\,.\,\phi\,u}{du}\sqrt{1+y'^2},\ T_1=1,\ T_2=0,\ \&c.$$

Hence the equation K=0, or

$$\lambda\,T-\frac{d\,(\lambda\,T_1)}{dx}+\frac{d^2\,(\lambda\,T_2)}{dx^2}-\&c.=0,$$

from which λ is to be determined, becomes

$$2a\lambda\frac{d.\phi u}{du}\sqrt{1+y'^2}-\frac{d\lambda}{dx}=0,$$

whence,

$$\frac{d\lambda}{\lambda}=2a\frac{d.\phi u}{du}\sqrt{1+y'^2}\,dx;$$

$$\therefore \lambda=c\,e^{\int 2a\frac{d.\phi u}{du}\sqrt{1+y'^2}\,dx},$$

c being an arbitrary constant quantity, and e the base of the Naperian logarithms.

Again the equation $H=0$ becomes simply

$$\frac{d(\lambda P)}{dx}=0;$$

whence by replacing P and λ by their values, and integrating, we have

$$\frac{2a\phi u y}{\sqrt{1+y'^2}}e^{\int 2a\frac{d.\phi u}{du}\sqrt{1+y'^2}\,dx}=C;$$

C being another arbitrary constant including the former.

From this equation we have

$$\int 2a\frac{d.\phi u}{dx}\sqrt{1+y'^2}\,dx=\log\left(\frac{C}{2a}\cdot\frac{\sqrt{1+y'^2}}{\phi u y'}\right);$$

and, differentiating,

$$2\,a\,\frac{d\,.\,\phi\,u}{d\,x}\sqrt{1+y'^2}\,dx=\frac{y'\,d\,y'}{1+y'^2}-\frac{d\,\phi\,u}{\phi\,u}-\frac{d\,y'}{y'}$$

$$=-\frac{d\,y'}{y'\,(1+y'^{\,2}}-\frac{d\,.\,\phi\,u}{\phi\,u};$$

from which $d\,u$ must be eliminated by means of the given equation,

$$\frac{d\,u}{d\,x}-2\,g+2\,a\,\phi\,u\sqrt{1+y'^2}=0.$$

Let $\phi\,u=u^n=v^{2\,n}$, then $\frac{d\,.\,\phi\,u}{d\,u}=n\,u^{n-1}$,

$\frac{d\,.\,\phi\,u}{\phi\,u}=\frac{n\,d\,u}{u}$; and we shall have

$$2\,a\,n\,u^n\sqrt{1+y'^2}dx+\frac{u\,d\,y'}{y'\,(1+y'^2)}+n\,d\,u=0.$$

Multiply the equation

$$d\,u-2\,g\,d\,x+2\,a\,u^n\sqrt{1+y^2}\,d\,x=0,$$

by n, and subtract it from the preceding, then we shall have

$$\frac{u\,d\,y'}{y\,(1+y'^2)}+2\,n\,g\,d\,x=0,$$

$$\therefore u=-\frac{2\,n\,g\,y'\,(1+y'^2)}{y''};$$

whence by substitution in the proposed equation, we obtain the differential equation to the curve.

We may otherwise arrive at the differential equation, by means of the equations,

$$du - 2g\,dx + 2a\,\phi u\sqrt{1+y'^2}\,dx = 0 \ldots (1.)$$

$$\frac{d\lambda}{dx} = 2a\lambda\frac{d.\phi u}{du}\sqrt{1+y^2} \ldots\ldots\ldots\ldots\ldots\ldots (2.)$$

$$\frac{2a\lambda\phi u y'}{\sqrt{1+y'^2}} = C \ldots\ldots\ldots\ldots\ldots\ldots\ldots (3.)$$

Eliminate $d\,u$ from equ. (2) by means of equ. (1), and we have

$$\begin{aligned}
2g\,d\lambda &= 2a\sqrt{1+y'^2}\,(\lambda\,d.\phi\,u + \phi\,u\,d\lambda) \\
&= 2a\sqrt{1+y'^2}\,d\,(\lambda\,\phi\,u) \\
&= 2a\,d\,(\lambda\,\phi\,u\sqrt{1+y'^2}) - \frac{2a\lambda\,\phi\,u\,y'\,dy'}{\sqrt{1+y'^2}} \\
&= d\left(\frac{C(1+y'^2)}{y'}\right) - C\,dy', \text{ by equ. (3);}
\end{aligned}$$

integrating,

$$\begin{aligned}
2g\lambda &= \frac{C(1+y'^2)}{y'} - C\,y' + C \\
&= \frac{C}{y'} + C';
\end{aligned}$$

C' being an arbitrary constant.

Hence equ. (3), by the elimination of λ, becomes

$$\frac{a\,\phi\,u}{g\sqrt{1+y'^2}}(C+C'y')=C,$$

which is the differential equation to the required curve.

With respect to the equation of limits, we must observe that $V=\frac{d\,u}{d\,x}-2g+2\,a\phi u\sqrt{1+y'^2}$, and since u is to be a maximum, the integral $\int V_{,}\,d\,x$ corresponds to $\int\frac{d\,u}{d\,x}\,d\,x$; hence the equation $\Delta=0$, becomes

$$\left(\lambda\frac{d\,u}{d\,x}-2g\,\lambda+2\,a\,\lambda\,\varphi\,u\sqrt{1+y'^2}+\frac{d\,u}{d\,x}\right)\delta\,x$$

$$+\frac{2\,a\,\lambda\,g\,\phi\,u}{\sqrt{1+y'^2}}\omega+(\lambda+1)\,\Omega=0;$$

since $\Gamma=(\lambda\,V+V_{,})\,\delta\,x+\lambda\,P\,\omega$

$$=\left(\lambda\,\frac{d\,u}{d\,x}-2g\,\lambda+2\,a\,\lambda\,\varphi\,u\sqrt{1+y'^2}+\frac{d\,u}{d\,x}\right)\delta\,x$$

$$+\frac{2\,a\,\lambda\,g\,\phi\,u}{\sqrt{1+y'^2}}\,\omega;$$

$$\mathrm{E}=(\lambda\,T_{,}+\pi)\Omega=(\lambda+1)\Omega.$$

Restoring the values of ω and Ω, putting for $2g\lambda$ and $2a\lambda\phi u$ their values

$$\frac{C}{y'}+C \text{ and } \frac{C\sqrt{1+y'^2}}{y'},$$

and reducing, we obtain

$$-C'\,\delta x+C\,\delta y+(\lambda+1)\,\delta u=0.$$

If we suppose the two limits of the curve to be independent of each other, we shall have

$$\Delta_0=0, \quad \Delta_1=0;$$

hence,

$$-C'\,\delta x_0+C\,\delta y_0+(\lambda_0+1)\,\delta u_0=0,$$
$$-C'\,\delta x_1+C\,\delta y_1+(\lambda_1+1)\,\delta u_1=0.$$

The initial velocity v_0, the square of which is u_0, must be given. If we suppose it independent of the point of departure, u_0 will be a constant quantity, and consequently its variation δu_0 equal to 0. With respect to δu_1, since it is entirely indeterminate, we may assume its coefficient $\lambda_1+1=0$, or $\lambda_1=-1$; whence we may obtain the value of the arbitrary constant C'. Thus the two equations of limits are reduced to

$$-C'\,\delta x_0+C\,\delta y_0=0,$$
$$-C'\,\delta x_1+C\,\delta y_1=0.$$

If the limits or extreme points of the curve be given, δx_0, δy_0, δx_1, δy_1, vanish, and the preceding equations verify themselves.

But if the extremities of the required curve be situated upon two given curves, and the motion be supposed to commence at the first curve; representing its equation by $\Phi(x, y) = 0$, we deduce

$$\frac{d\cdot\Phi(x,y)}{d\,x}\,\delta x + \frac{d\cdot\Phi(x,y)}{d\,x}\,\delta y = 0,$$

$$\frac{d\cdot\Phi(x,y)}{d\,x} + \frac{d\cdot\Phi(x,y)}{d\,x}\,y' = 0,$$

from which we have

$$\delta y - y'\,\delta x = 0';$$

whence at the limit,

$$\delta y_0 - y'_0\,\delta x_0 = 0.$$

Similarly, at the second limit,

$$\delta y_1 - y'_1\,\delta x_1 = 0:$$

these, combined with the equations of limits, give

$$y'_0 = \frac{C'}{C},\ y'_1 = \frac{C'}{C}.$$

Now y'_0, y'_1 represent, respectively, the trigonometrical tangents of the angles which the tangents to the two given curves at the limits make with the axis of x; whence it appears that the tangents at these points are parallel to one another, as in the curve of quickest descent in a non-resisting medium. (Section V. Example VII.) p. 80

EXAMPLE II.

Required the curve of quickest descent in a resisting medium.

Let u represent the square of the velocity, and suppose $\varphi\, u$ to be the function of the velocity which is proportional to the resistance; then we shall have, as in the preceding example, the equation of condition,

$$\frac{d\,u}{d\,x}-2\,g+2\,a\,\varphi\,u\sqrt{1+y'^2}=0,$$

for the determination of u, and the time, which must be a minimum, is expressed by

$$\int\sqrt{\frac{1+y'^2}{u}}\,d\,x.$$

Whence,

$$V=\frac{d\,u}{d\,x}-2\,g+2\,a\,\phi\,u\sqrt{1+y'^2}=0,$$

$$V_{,}=\sqrt{\frac{1+y'^2}{u}};$$

$$d\,V=d\,u'+2\,a\sqrt{1+y'^2}\,\frac{d\cdot\phi\,u}{d\,u}\,du+\frac{2\,a\,\phi\,u\,y'\,d\,y'}{\sqrt{1+y'^2}},$$

$$d\,V_{,}=\frac{y'\,d\,y'}{\sqrt{u}\,\sqrt{1+y'^2}}-\frac{\sqrt{1+y'^2}}{2\,u^{\frac{3}{2}}}\,d\,u.$$

Therefore,

$$M=0,\ \ N=0,\ \ P=\frac{2\,a\,\phi\,u\,y'}{\sqrt{1+y'^2}},\ Q=0,\ \&c.,$$

$$T=2\,a\sqrt{1+y'^2}\,\frac{d\cdot\phi\,u}{d\,u},\ T_1=1,\ \ T_2=0,\ \&c.$$

$$M_{,}=0,\ N_{,}=0,\ P_{,}=\frac{y}{\sqrt{u}\,\sqrt{1+y'^2}},\ Q_{,}=0,\ \&c.$$

$$\nu=-\frac{\sqrt{1+y'^2}}{2\,u^{\frac{3}{2}}},\ \pi=0,\ \&c.,$$

and the equations

$$\mathrm{H}=0,\qquad \mathrm{K}=0,$$

become,

$$\frac{d\left(\frac{y}{\sqrt{u}\sqrt{1+y'^2}}\right)}{dx}+\frac{d\left(\frac{2a\lambda\phi u y'}{\sqrt{1+y'^2}}\right)}{dx}=0,$$

$$-\frac{\sqrt{1+y'^2}}{2u^{\frac{3}{2}}}+2a\lambda\sqrt{1+y'^2}\,\frac{d.\phi u}{du}-\frac{d\lambda}{dx}=0.$$

If we make $t=\frac{1}{\sqrt{u}}+2a\lambda\phi u$, the first equation, by integration, gives

$$\frac{ty'}{\sqrt{1+y'^2}}=C\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1),$$

C being an arbitrary constant: the second may be written thus,

$$\frac{d\lambda}{dx}-\left(2a\lambda\frac{d.\phi u}{du}-\frac{1}{2u^{\frac{3}{2}}}\right)\sqrt{1+y'^2}=0\ldots(2.),$$

and the equation of condition,

$$u'-2g+2a\varphi u\sqrt{1+y'^2}=0\ \ldots\ldots\ldots\ldots\ (3.)$$

furnishes a third, from which the equation to the curve is to be derived.

From the assumed value of t we have

$$\frac{dt}{du}=-\frac{1}{2u^{\frac{3}{2}}}+2a\phi u\frac{d\lambda}{du}+2a\lambda\frac{d\cdot\phi u}{du};$$

hence the equation (2.) by substitution becomes

$$\frac{d\lambda}{dx}+\left(2a\phi u\frac{d\lambda}{du}-\frac{dt}{du}\right)\sqrt{1+y^2}=0;$$

eliminating du by means of equ. (3), we obtain

$$2g\frac{d\lambda}{dx}-\frac{dt}{dx}\sqrt{1+y'^2}=0;$$

and since

$$\frac{dt}{dx}\sqrt{1+y'^2}\,dx=d\left(t\sqrt{1+y'^2}\right)-\frac{ty'\,dy'}{\sqrt{1+y'^2}}$$

$=d(t\sqrt{1+y'^2})-C\,dy'$, we have,

$$2g\frac{d\lambda}{dx}-\frac{d(t\sqrt{1+y'^2})}{dx}+Cy''=0:$$

integrating,

$$2g\lambda-t\sqrt{1+y'^2}+Cy'=C';$$

C' being another arbitrary constant.

Substituting for t its value in equ. (1), we find

$$\lambda=\frac{\frac{C}{y'}+C'}{2g};$$

hence the expression for t may be written

$$t=\frac{1}{\sqrt{u}}+\frac{a\,\phi\,u}{g}\left(\frac{C}{y'}+C'\right);$$

and by substitution in equ. (1), we have

$$\frac{1}{\sqrt{u}}=\frac{C\sqrt{1+y'^2}}{y'}-\frac{a\,\phi\,u}{g}\left(\frac{C}{y'}+C'\right).$$

The value of u and $d\,u$ deduced from this and substituted in the equation of condition, will produce a differential equation of the second order, which is the equation to the required curve.

In the case of a vacuum, or non-resisting medium, $a\,\phi\,u=0$; hence we have

$$\frac{1}{\sqrt{u}}=\frac{C\sqrt{1+y'^2}}{y'},$$

or

$$\frac{y'}{\sqrt{u}\sqrt{1+y'^2}}=C\ldots\ldots(4.)$$

Also the equation of condition becomes

$$\frac{d\,u}{d\,x}-2\,g=0,$$

$$\therefore u=2gx+b;$$

b being an arbitrary constant.

At the first limit we have

$$u_0=2gx_0+b;$$

eliminating b, we shall have for the complete value of u,

$$u=2g(x-x_0)+u_0.$$

Hence by substituting and writing $\frac{1}{c}$ for C^2 in equ. (4), we obtain

$$y'=\sqrt{\frac{u}{c-u}};$$

Now this equation being of the same form when the value of u in terms of x is substituted, the curve is the common cycloid.

With respect to the equations of limits, if we make in the value of Δ or $\Gamma+E$, the necessary substitutions, the equation $\Delta=0$ will become

$$\left.\begin{array}{l}\dfrac{\sqrt{1+y'^2}}{\sqrt{u}}\delta x+\left(\dfrac{1}{\sqrt{u}}+2a\lambda\phi u\right)\dfrac{y'(\delta y-y'\delta x)}{\sqrt{1+y'^2}}\\ \qquad\qquad\qquad\qquad+\lambda(\delta u-u'\delta x)\end{array}\right\}=0;$$

replacing u' by its value $2g-2a\phi u\sqrt{1+y'^2}$, writing t for $\frac{1}{\sqrt{u}}+2\,a\,\lambda\,\varphi\,u$, and simplifying by means of equ. (1), the preceding equation is reduced to

$$\left(\frac{C}{y'}-2\,g\,\lambda\right)\delta\,x+C\,\delta\,y+\lambda\,\delta\,u=0.$$

Since u represents the square of the velocity, and the equation of condition being of the first order with respect to u, the value of u deduced by integration will contain an arbitrary constant which will depend upon the initial velocity impressed upon the body. We may therefore regard the value of u at the first limit as a given function of the initial coordinates x_0 and y_0. Representing this function by $u_0=\Psi(x_0, y_0)$, we have

$$\delta\,u_0=\frac{d\,.\,\Psi\,(x_0,\,y_0)}{d\,x_0}\,\delta\,x_0+\frac{d\,.\,\Psi\,(x_0,\,y_0)}{d\,y_0}\,\delta\,y_0,$$

so that the equation $\Delta=0$, gives

$$\left(\frac{C}{y'_0}-2\,g\,\lambda_0+\lambda_0\frac{d\,.\,\Psi\,(x_0,\,y_0)}{d\,x_0}\right)\delta\,x_0$$

$$+\left(C+\lambda_0\frac{d\,.\,\Psi\,(x_0,\,y_0)}{d\,y_0}\right)\delta\,y_0=0;$$

which is the equation of the first limit.

The case is not the same with respect to u at the second limit; for the variation δu_1, in the equation $\Delta_1=0$, is entirely indeterminate. It is necessary then to equate to 0 its coefficient λ_1, which will enable us to determine the value of the constant C': thus we shall have

$$\frac{C}{y'_1}+C'=\lambda_1=0;\ \text{whence}\ C'=-\frac{C}{y'_1},$$

so that the complete expression for λ will be

$$\lambda=\frac{C}{2g}\left(\frac{1}{y'}-\frac{1}{y_1}\right),$$

therefore we have,

$$2g\,\lambda_0=C\left(\frac{1}{y'_0}-\frac{1}{y'_1}\right),$$

which it will be necessary to substitute in the equation of the first limit.

The equation of the second limit $\Delta_1=0$, since $\lambda_1=0$, becomes

$$\frac{\delta x_1}{y'_1}+\delta y_1=0,\ \text{or,}\ 1+y'_1\frac{\delta y_1}{\delta x_1}=0.$$

Let $dy=n\,dx$ be the equation to the given curve at the second limit, then we shall have, at this limit,

$$\frac{dy_1}{dx_1}\ \text{or}\ \frac{\delta y_1}{\delta x_1}=n_1,$$

and the above equation may be replaced by

$$1+n_1\frac{dy_1}{dx_1}=0:$$

whence we conclude that the curve of quickest descent cuts the curve which forms the second limit at right angles, a property which is independent of the law of resistance.

Returning to the equation of the first limit: by the substitution of the value of λ_0 found above, we shall have,

$$\left.\begin{array}{l}\left\{\dfrac{1}{y'_1}+\dfrac{1}{2g}\cdot\dfrac{d\,.\,\Psi(x_0,y_0)}{dx_0}\left(\dfrac{1}{y'_0}-\dfrac{1}{y'_1}\right)\right\}\delta x_0\\ +\left\{1+\dfrac{1}{2g}\cdot\dfrac{d\,.\,\Psi(x_0,y_0)}{dy_0}\left(\dfrac{1}{y'_0}-\dfrac{1}{y'_1}\right)\right\}\delta y_0\end{array}\right\}=0.$$

Now if we represent by $dy=m\,dx$ the differential equation to the curve forming the locus of the first limit, we shall have $\delta y_0=m_0\,\delta x_0$; whence, by substitution, and omitting the common factor δx_0, the preceding equation becomes

$$\left.\begin{array}{l}\dfrac{1}{y'_1}+\dfrac{1}{2g}\cdot\dfrac{d\,.\,\Psi(x_0,y_0)}{dx_0}\left(\dfrac{1}{y'_0}-\dfrac{1}{y'_1}\right)\\ +\left(1+\dfrac{1}{2g}\cdot\dfrac{d\,.\,\Psi(x_0,y_0)}{dy_0}\right)\left(\dfrac{1}{y'_0}-\dfrac{1}{y'_1}\right)m_0\end{array}\right\}=0.$$

If we suppose the initial velocity $\sqrt{u_0}$ to be independent of the point of departure, so that u_0 or $\Psi(x_0, y_0) =$ constant, we shall have

$$\frac{1}{y'_1}+m_0=0 \text{ or } 1+m_0\frac{dy_1}{dx_1}=0.$$

Now m_0 represents the trigonometrical tangent of the angle which the tangent to the curve of the first limit makes with the axis of x; and $\frac{dy_1}{dx_1}$ expresses the trigonometrical tangent of the angle which the tangent to the line of quickest descent, at the point where it meets the second limit, makes with the same axis; therefore the curve of the first limit is perpendicular to the line of quickest descent at the second limit; and as we have seen that the same line is perpendicular to the curve of the second limit, it follows that the line of quickest descent must meet the two given curves in points where their tangents are parallel to one another.

If we suppose the initial velocity to be that which the body would acquire in falling from a given point; calling h the height of this point above the horizontal axis of y, we shall have $h+x_0$ for the height due to the initial velocity $\sqrt{u_0}$. By the principles of Mechanics, we have

$$u_0=2g\,(h+x_0);$$

whence

$$\Psi(x_0, y_0) = 2g(h + x_0),$$

and consequently,

$$\frac{d.\Psi(x_0, y_0)}{dx_0} = 2g, \quad \frac{d\cdot\Psi(x_0, y_0)}{dy_0} = 0;$$

these reduce the equation of the first limit to

$$1 + m_0 \frac{dy_0}{dx_0} = 0;$$

which shews that the curve of quickest descent cuts the curve of the first limit also at right angles.

Thus the conditions relative to the limits are independent of the resistance of the medium. *Calcul des Fonctions*, p. 498.

SECTION IX.

40. HITHERTO we have been considering functions of only one independent variable; but there are questions in which one variable is a function of other variables not necessarily connected, for instance when $z = \phi(x, y)$ is the equation to a curve surface. If we wish to pass from this surface to another surface indefinitely near the first, the ordinate z may vary independently of x and y, or a point in one surface may be transported to the other surface in the direction of the ordinate z. But, for generality, we shall consider the variation of z as a function of x and y. In whatever way the double integral $\int\int V\,dx\,dy$, may be supposed to arise, it may always be referred to a volume or to a curve surface; and the simplest way is to consider the limits as determined by cylindrical surfaces perpendicular to the plane xy, in which case, δx, δy vanish at the limits. When the limits are the intersections of the required surface with surfaces inclined to the plane xy, these co-ordinates, as well

as z, vary at the limits. In general, however, the variation of x depends only upon x, and that of y only upon y.

In fact let

$$dz = p\,dx + p_{,}\,dy,$$

$$dp = q\,dx + q_{,}\,dy, \quad dp_{,} = q_{,}\,dx + q_{,,}\,dy,$$

&c.

then,

$$\delta p = \delta\frac{dz}{dx} = \frac{d\,\delta z - p\,d\,\delta x}{dx},$$

$$\delta p_{,} = \frac{dz}{dy} = \frac{d\,\delta z - p_{,}\,d\,\delta y}{dy},$$

where it must be observed that y is considered constant in the first, and x in the second.

Now,

$$q_{,} = \frac{dp}{dy} \text{ or } = \frac{dp_{,}}{dx};$$

$$\therefore \delta q_{,} = \delta\frac{dp}{dy} = \frac{d\,\delta p - q_{,}\,d\,\delta y}{dy},$$

$$\text{or} = \frac{d\,\delta p_{,} - q_{,}\,d\,\delta x}{dx}:$$

but

$$\frac{d\,\delta p}{d y}=\frac{d\left(\frac{d\,\delta z-p\,d\,\delta x}{d x}\right)}{d y}$$

$$=\frac{d^2\,\delta z}{dx\,dy}-\frac{q_{,}\,d\,\delta x}{d x}-\frac{p\,d^2\,\delta x}{d x\,d y};$$

and

$$\frac{d\,\delta p_{,}}{d x}=\frac{d\left(\frac{d\,\delta z-p_{,}\,d\,\delta y}{d y}\right)}{d x}$$

$$=\frac{d^2\,\delta z}{dy\,dx}-\frac{q_{,}\,d\,\delta y}{d y}-\frac{p_{,}\,d^2\,\delta y}{d y\,d x}:$$

whence we have, for the value of $\delta q_{,}$, the two quantities,

$$\frac{d^2\,\delta z}{dx\,dy}-\frac{q_{,}\,d\,\delta y}{d y}-\frac{q_{,}\,d\,\delta x}{d x}-\frac{p\,d^2\,\delta x}{dx\,dy},$$

$$\frac{d^2\,\delta z}{dy\,dx}-\frac{q_{,}\,d\,\delta x}{d x}-\frac{q_{,}\,d\,\delta y}{d y}-\frac{p_{,}\,d^2\,\delta y}{dy\,dx};$$

in which we find the two essentially different terms

$$\frac{p\,d^2\,\delta x}{d x\,d y}\text{ and }\frac{p_{,}\,d^2\,\delta y}{d y\,d x};$$

these can only be made to disappear by supposing y not to enter into δx, nor x into δy.

41. Let K be a given function of the two independent variables x, y and other quantities depending upon them; we shall represent by K', K'', K''', &c., the differential coefficients of K taken with respect to x and quantities depending upon x, and by $K_{,}$, $K_{,,}$, $K_{,,,}$, &c., the analogous differential coefficients relative to y. Thus we shall have

$$\frac{d\,(K)}{d\,x} = K', \quad \frac{d\,(K)}{d\,y} = K_{,}, \quad \frac{d^2\,(K)}{d\,x^2} = K'',$$

$$\frac{d^2\,(K)}{d\,x\,dy} = K'_{,}, \quad \frac{d^3\,(K)}{d\,x\,d\,y^2} = K_{,,}, \text{ \&c.}$$

The limits, whether known or unknown, of the double integral $\iint K\,d\,x\,d\,y$, will not be indicated here; but we may observe that if this integral belong to a zone of surface comprised between two curves returning into themselves, either of simple or of double curvature, and x, y, z be the rectangular co-ordinates of any point of the surface; the two limits correspond to the projections of these two curves upon the plane $x\,y$; and we shall represent by $\left(\int K\,d\,x\right)_0$, $\left(\int K\,d\,y\right)_0$, the values of any integrals $\int K\,dx, \int K\,dy$ at the first limit, and by $\left(\int K\,dx\right)_1, \left(\int K\,dy\right)_1$ their values at the second limit; so that the first two belong to the interior curve and extend to its entire outline, and the latter two extend to the entire length of the exterior curve.

42. Now z being an unknown function of x and y, also V a given function of

$$x,\ y,\ z,\ z',\ z_{,},\ z'',\ z'_{,},\ z_{,,}\ \&c.,$$

we proceed to investigate the variation of the double integral

$$\mathrm{U} = \iint V\,dx\,dy.$$

We have at first,

$$\begin{aligned}\delta\mathrm{U} &= \iint \delta\,(V\,dx\,dy)\\ &= \iint \delta\,V\,dx\,dy + \iint V\,\delta\,(dx\,dy)\\ &= \iint \delta\,V dx\,dy + \iint V\,(dy\,\delta\,dx + dx\,\delta dy);\end{aligned}$$

integrating by parts,

$$\begin{aligned}&\iint V\,dy\,\delta\,dx, \text{ or} \iint V\,dy\,d\,\delta\,x\\ &= \int \delta\,x\ V\,dy - \iint \delta\,x\,V'\,dx\,dy,\end{aligned}$$

$$\iint V\,dx\,\delta\,dy = \int \delta y\,V\,dx - \iint \delta y\,V_{,}\,dx\,dy;$$

whence we have

$$\delta\,\mathrm{U} = \iint \delta\,V\,dy\,dx + \int \delta x\,V\,dy + \int \delta y\,V\,dx$$

$$-\left(\iint V'\delta x + \iint V_{,}\,\delta y\right) dx\,dy.$$

$$= \int \delta x\,V\,dy + \int \delta y\,V\,dx$$

$$+\iint\left(\delta V - V'\,\delta x - V_{,}\,\delta y\right) dx\,dy \ldots\ldots(1)$$

Representing the complete differential of V by

$$d\,V = L\,dx + M\,dy + N\,dz + P\,dz' + Q\,dz_{,}$$
$$+ R\,dz'' + S\,dz'_{,} + T\,dz_{,,} + \&c.,$$

we have

$$V' = L + N\,z' + P\,z'' + Q\,z'_{,} + \&c.$$
$$+ R\,z''' + S\,z''_{,} + \&c.,$$
$$+ T\,z'_{,,} + \&c.,$$

$$V_{,} = M + N\,z_{,} + P\,z'_{,} + Q\,z_{,,} + \&c.$$
$$+ R\,z''_{,} + S\,z'_{,,} + \&c.$$
$$+ T\,z_{,,,} + \&c.,$$

$$\delta\,V = L\,\delta x + M\,\delta y + N\,\delta z + P\,\delta z' + Q\,\delta z_{,}$$
$$+ R\,\delta z' + S\,\delta z'_{,} + T\,\delta z_{,,}\ \&c.$$

These values being substituted in the function $\delta V - V' \delta x - V_{,} \delta y$, give, after some reductions, the equivalent expression

$$N\,(\delta z - z' \delta x - z_{,} \delta y) + P\,(\delta z' - z'' \delta x - z'_{,} \delta y)$$
$$+ Q\,(\delta z_{,} - z'_{,} \delta x - z_{,,} \delta y) + R\,(\delta z'' - z''' \delta x - z''_{,} \delta y)$$
$$+ S\,(\delta z'_{,} - z''_{,} \delta x - z'_{,,} \delta y) + T(\delta z_{,,} - z'_{,,} \delta x - z_{,,,} \delta y)$$
$$+ \&c.$$

Since

$$z' = \frac{d\,z}{d\,x},\ \delta z' = \delta \frac{d\,z}{d\,x}$$
$$= \frac{d\,\delta z - z'\,d\,\delta x}{d\,x};$$

$$\therefore \delta z' - z'' \delta x - z'_{,} \delta y = \frac{d\,(\delta z - z' \delta x - z_{,} \delta y)}{d\,x}.$$

Also $z_{,} = \frac{d\,z}{d\,y}$, whence $\delta z_{,} = \delta \frac{d\,z}{d\,y}$

$$= \frac{d\,\delta z - z_{,}\,d\,\delta y}{d\,y};$$

$$\therefore \delta z_{,} - z'_{,} \delta x - z_{,,} \delta y = \frac{d\,(\delta z - z' \delta x - z_{,} \delta y)}{d\,x}.$$

If, for brevity, we make

$$\delta z - z' \,\delta x - z_{,} \,\delta y = \omega,$$

the preceding equations may be written

$$\left.\begin{array}{l} \delta z' - z'' \,\delta x - z'_{,} \,\delta y = \omega' \\ \delta z_{,} - z'_{,} \,\delta x - z_{,,} \,\delta y = \omega_{,} \end{array}\right\} \ldots (2).$$

Now since these exist for any function z of x, y, we may substitute successively z', $z_{,}$, z'', &c., in the place of z. Hence if z' be written for z in the first equation, it becomes

$$\delta z'' - z''' \,\delta x - z''_{,} \,\delta y = \frac{d\,(\delta z' - z'' \,\delta x - z'_{,} \,\delta y)}{d\,x};$$

but differentiating the first equation (2), with respect to x, we have

$$\frac{d\,(\delta z' - z'' \,\delta x - z'_{,} \,\delta y)}{d\,x} = \frac{d\,\omega}{d\,x};$$

$$\therefore\ \delta z'' - z''' \,\delta x - z''_{,} \,\delta y = \omega''.$$

By writing $z_{,}$ for z in the second equation, we shall find in the same manner,

$$\delta z_{,,} - z'_{,,} \,\delta x - z_{,,,} \,\delta y = \omega_{,,}.$$

Again, if we put z' in the place of z in the second equation, we shall have

$$\delta z'_{,} - z''_{,}\,\delta x - z'_{,,}\,\delta y = \frac{d\,(\delta z' - z''\,\delta x - z'_{,}\,\delta y)}{d\,y};$$

and the differentiation of the first equation (2), relative to y, gives

$$\frac{d\,(\delta z' - z''\,\delta x - z'_{,}\,\delta y)}{d\,y} = \frac{d\,\omega'}{d\,y};$$

whence we have

$$\delta z'_{,} - z''_{,}\,\delta x - z'_{,,}\,\delta y = \omega'_{,};$$

continuing in this manner it will be easy to see, whatever be the indices m and n, that

$$\delta z^{(m)}_{(n)} - z^{(m+1)}_{(n)}\,\delta x - z^{(m)}_{(n+1)}\,\delta y = \omega^{(m)}_{(n)}.$$

By means of this general theorem the preceding value of

$$\delta V - V'\,\delta x - V_{,}\,\delta y,$$

is reduced to

$$N\omega + P\omega' + Q\omega_{,} + R\,\omega'' + S\,\omega'_{,} + T\,\omega_{,,} + \&c.$$

Hence the expression for the variation of U, equ. (1), becomes

$$\delta \mathrm{U} = \int V\,\delta x\;dy + \int V\,\delta y\;dx$$

$$+\iint\left(N\omega + P\omega' + Q\omega_{,} + R\omega'' + S\omega'_{,} + \&c.\right)dxdy \quad (3).$$

If we integrate by parts, we shall make the differential coefficients of ω disappear under the double sign $\iint$. Thus we have, at first,

$$\iint P\,\omega'\,dx\,dy = \int P\,\omega\,dy - \iint P'\,\omega\,dx\,dy,$$

$$\iint Q\,\omega_{,}\,dx\,dy = \int Q\,\omega\,dx - \iint Q_{,}\,\omega\,dx\,dy;$$

also, two successive integrations with respect to x and to y will give, respectively,

$$\iint R\omega''\,dx\,dy = \int R\omega'\,dy - \int R'\omega\,dy + \iint R''\omega\,dx\,dy,$$

$$\iint T\omega_{,,}\,dx\,dy = \int T\omega_{,}\,dx - \int T_{,}\omega\,dx + \iint T_{,,}\omega\,dx\,dy;$$

Again, integrating first with respect to y and then with respect to x, we shall have

$$\iint S\omega'_{,}\,dx\,dy = \int S\omega'\,dx - \int S\omega_{,}\,dy + \iint S'_{,}\omega\,dx\,dy;$$

but integrating in an inverse order, we have

$$\iint S\omega'_{,}\,dx\,dy = \int S\omega_{,}\,dy - \int S\omega'\,dx + \iint S'_{,}\,\omega\,dx\,dy;$$

for symmetry of expression in the variation of U, we shall take half the sum of these equivalent expressions; whence we have

$$\iint S\omega'_{,}\,dx\,dy = \tfrac{1}{2}\int S\omega'\,dx + \tfrac{1}{2}\int S\omega_{,}\,dy$$
$$-\tfrac{1}{2}\int S_{,}\omega\,dy - \tfrac{1}{2}\int S'\omega\,dx$$
$$+\iint S'_{,}\omega\,dx\,dy.$$

In a similar manner we may integrate the following terms, $\iint T\omega_{,,}\,dx\,dy$, &c.

Collecting these terms and substituting them in equation (3), taking the single integrals between limits, and supposing the double integrals to be taken between the same limits, we shall have for the variation sought,

$$\delta\,\mathrm{U} = \Gamma + \iint \mathrm{H}\,\omega\,dx\,dy \ldots\ldots (C),$$

making, for brevity,

$$\mathrm{H} = N - P' - Q_{,} + R'' + S'_{,} + T_{,,} - \&c.$$

$$\Gamma = \left(\int V\,\delta x\,dy\right)_1 - \left(\int V\,\delta x\,dy\right)_0$$
$$+\left(\int V\,\delta y\,dx\right)_1 - \left(\int V\,\delta y\,dx\right)_0$$
$$+\left(\int P\omega\,dy + \int Q\omega\,dx - \int R'\omega\,dy\right.$$
$$\left. - \tfrac{1}{2}\int S_{,}\omega\,dy - \tfrac{1}{2}\int S'\omega\,dx - \int T_{,}\omega\,dx + \&c.\right)_1$$

$$
\begin{aligned}
&-\Big(\int P\omega\,dy+\int Q\omega\,dx-\int R'\omega\,dy\\
&-\tfrac{1}{2}\int S_{,}\omega\,dy-\tfrac{1}{2}\int S'\omega\,dx-\int T_{,}\omega\,dx+\&c.\Big)_0\\
&+\Big(\int R\omega'\,dy+\tfrac{1}{2}\int S\omega'\,dx-\&c.\Big)_1\\
&-\Big(\int R\omega'\,dy+\tfrac{1}{2}\int S\omega'\,dx-\&c.\Big)_0\\
&+\Big(\int T\omega_{,}\,dx+\tfrac{1}{2}\int S\omega_{,}\,dy-\&c.\Big)_1\\
&-\Big(\int T\omega_{,}\,dx+\tfrac{1}{2}\int S\omega_{,}\,dy-\&c.\Big)_0\\
&+\&c.
\end{aligned}
$$

43. When U is a maximum or minimum, we must have $\delta U=0$; and the double integral which the formula (C) contains, being irreducible to single integrals, since ω is an arbitrary function of x and y, it is necessary that the two terms of that formula should separately vanish; whence we conclude that

$$\Gamma=0, \qquad H=0,$$

which are the equations of the maximum or minimum of the double integral proposed.

The equation $H=0$ will make known the function z in terms of x and y. If the given function V contain partial differentials of the order n, that equation will be of the order $2n$, and its complete integral will contain $2n$ arbitrary functions. If the limits of the double integral are variable and unknown, the equation $\Gamma=0$ will be decomposed into several others, the number and nature of which, according to different cases, involve considerations of much delicacy and difficulty. These we shall not examine; but the student who may be desirous of pursuing the subject, may consult a memoir of M. Poisson's, *Sur le Calcul des Variations* in the *Mém. de l'Acad. Roy. des Scien.* Tom. XII. pp. 298—320.

44. We may easily extend the preceding analysis to triple, quadruple, &c., integrals. In the case of a triple integral, we shall find for its variation an expression analogous to the formula (C), composed of a triple integral and of another part containing double integrals relative to the limits of the triple integral, which we are considering. We may also suppose the quantity under the signs of integration to contain several unknown functions of independent variables; these functions being independent of one another or connected by given partial differential

equations: the consideration of which embraces nothing new either in principle or in the mode of investigation.

45. As to the determination of relative maxima or minima of multiple integrals, we may reduce the subject to a problem of absolute maximum or minimum by the method of (28), which evidently subsists, whatever be the number of the independent variables. Thus, if the first of the double integrals,

$$\iint V\,dx\,dy, \iint T\,dx\,dy, \iint W\,dx\,dy, \text{\&c.},$$

is to be a maximum or minimum, and the others are to have given values, the question will be solved by the determination of the absolute maximum or minimum of another double integral, viz.

$$\mathrm{U} = \iint (V + a\,T + b\,W + \text{\&c.})\,dx\,dy,$$

a, b, &c., being unknown constant quantities, to be determined from the given values of the latter integrals.

We shall terminate this subject with the following examples relative to double integrals.

EXAMPLE I.

To find a solid whose surface between certain limits is a minimum.

Here we have

$$U=\iint\sqrt{1+z'^2+z_{,}^2}\,dx\,dy, \text{ a minimum};$$

$$V=\sqrt{1+z'^2+z_{,}^2}:$$

$$\therefore\ N=0,\ P=\frac{z'}{V},\ Q=\frac{z_{,}}{V},\ R=0, \text{ \&c.}$$

$$P'=\frac{dP}{dx}=\frac{(1+z_{,}^2)\,z''-z'\,z_{,}\,z'_{,}}{V^{\frac{3}{2}}},$$

$$Q_{,}=\frac{dQ}{dy}=\frac{(1+z'^2)\,z_{,,}-z'\,z_{,}\,z'_{,}}{V^{\frac{3}{2}}},$$

$$R'=0, \text{ \&c.}$$

Hence the equation H=0 becomes

$$(1+z_{,}^2)\,z''-2\,z'\,z_{,}\,z'_{,}+(1+z'^2)\,z_{,,}=0;$$

which is the differential equation to the solid of least surface.

EXAMPLE II.

Among all solids of equal capacity to find that which has the least surface.

Since the surface which is to be a minimum, is expressed by $\iint \sqrt{1+z'^2+z_,^2}\, dx\, dy$, and the given solidity by $\iint z\, dx\, dy$; we have to make the function

$$U=\iint\left(\sqrt{1+z'^2+z_,^2}+a z\right) dx\, dy, \text{ a minimum.}$$

Here

$$V=\sqrt{1+z'^2+z_,^2}+a\, z;$$

$$N=a,\ P'=\frac{(1+z_,^2)\, z''-z' z_, z'_,}{\sqrt{1+z'^2+z_,^2}},$$

$$Q_,=\frac{(1+z'^2)\, z_{,,}-z'\, z_, z'_,}{\sqrt{1+z'^2+z_,^2}},\ R''=0, \text{ \&c.}$$

hence we have

$$a(1+z'^2+z_,^2)^{\frac{3}{2}}-(1+z'^2)z''+2z' z_, z'_,-(1+z_,^2)z_{,,}=0,$$

the general equation to the curve surface.

The complete integral of each of these equations will contain two arbitrary functions, which must be assumed in such a way that the surface may pass through the given curves which form the bounding limits.

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